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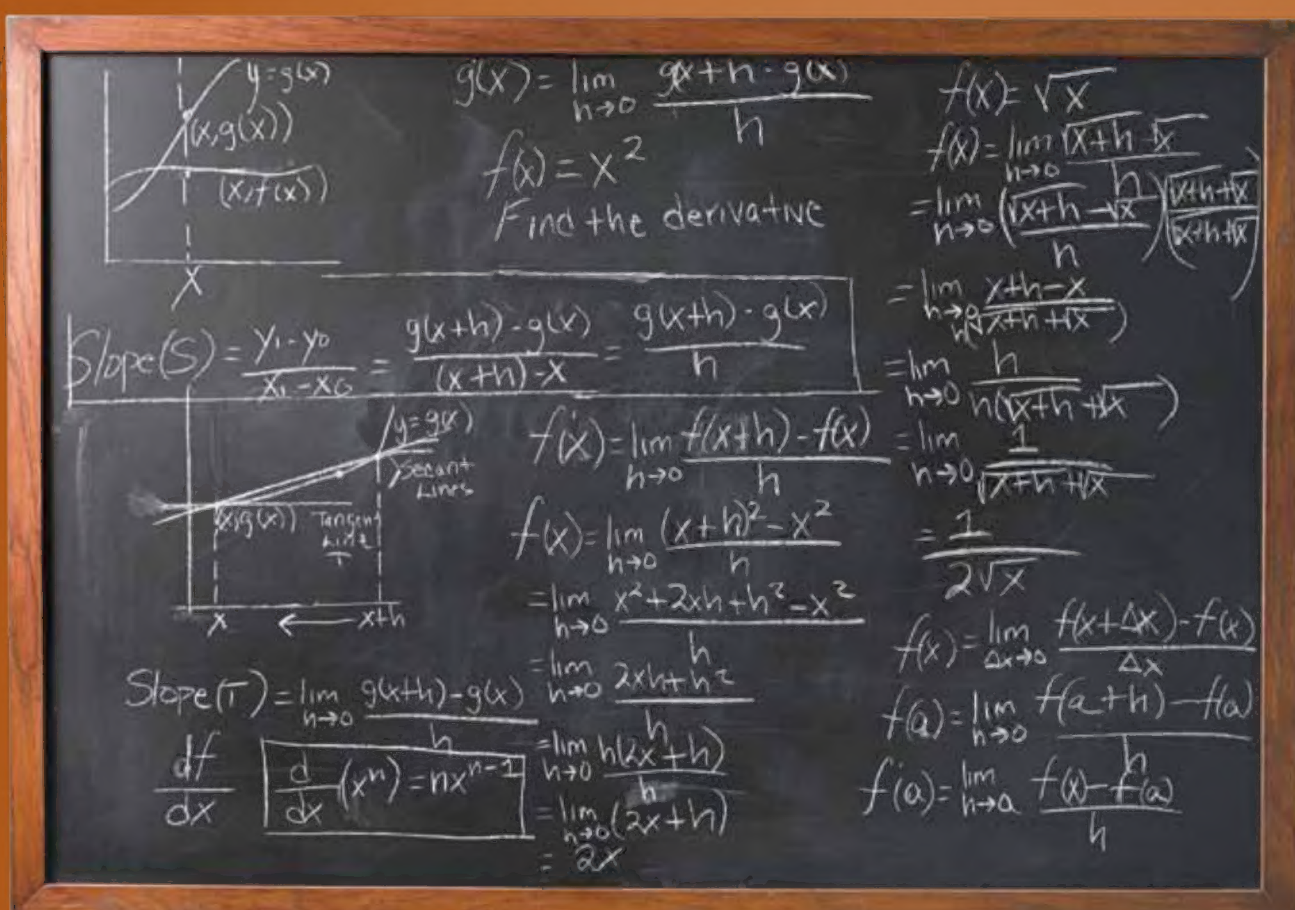
Understanding Calculus: Problems, Solutions, and Tips

Course Workbook

Professor Bruce H. Edwards
University of Florida



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www.thegreatcourses.com

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Bruce H. Edwards, Ph.D.

Professor of Mathematics, University of Florida

Bruce H. Edwards has been a Professor of Mathematics at the University of Florida since 1976. He received his B.S. in Mathematics from Stanford University in 1968 and his Ph.D. in Mathematics from Dartmouth College in 1976. From 1968 to 1972, he was a Peace Corps volunteer in Colombia, where he taught mathematics (in Spanish) near Bogotá, at La Universidad Pedagógica y Tecnológica de Colombia.

Professor Edwards's early research interests were in the broad area of pure mathematics called algebra. His dissertation in quadratic forms was titled "Induction Techniques and Periodicity in Clifford Algebras." Beginning in 1978, he became interested in applied mathematics while working summers for NASA at the Langley Research Center in Virginia. This led to his research in the area of numerical analysis and the solution of differential equations. During his sabbatical year 1984–1985, he worked on 2-point boundary value problems with Professor Leo Xanthis at the Polytechnic of Central London. Professor Edwards's current research is focused on the algorithm called CORDIC that is used in computers and graphing calculators for calculating function values.

Professor Edwards has coauthored a wide range of mathematics textbooks with Professor Ron Larson of Penn State Erie, The Behrend College. They have published leading texts in the areas of calculus, applied calculus, linear algebra, finite mathematics, algebra, trigonometry, and precalculus. This course is based on the bestselling textbook *Calculus* (9^{th} edition, Brooks/Cole, 2010).

Professor Edwards has won many teaching awards at the University of Florida. He was named Teacher of the Year in the College of Liberal Arts and Sciences in 1979, 1981, and 1990. He was both the Liberal Arts and Sciences Student Council Teacher of the Year and the University of Florida Honors Program Teacher of the Year in 1990. He was also selected by the alumni affairs office to be the Distinguished Alumni Professor for 1991–1993. The winners of this 2-year award are selected by graduates of the university. The Florida Section of the Mathematical Association of America awarded him the Distinguished Service Award in 1995 for his work in mathematics education for the state of Florida. Finally, his textbooks have been honored with various awards from the Text and Academic Authors Association.

Professor Edwards has been a frequent speaker at both research conferences and meetings of the National Council of Teachers of Mathematics. He has spoken on issues relating to the Advanced Placement calculus examination, especially the use of graphing calculators.

Professor Edwards has taught a wide range of mathematics courses at the University of Florida, from first-year calculus to graduate-level classes in algebra and numerical analysis. He particularly enjoys teaching calculus to freshman, due to the beauty of the subject and the enthusiasm of the students.

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Understanding Calculus: Problems, Solutions, and Tips

Scope:

The goal of this course is for you to understand and appreciate the beautiful subject of calculus. You will see how calculus plays a fundamental role in all of science and engineering, as well as business and economics. You will learn about the 2 major ideas of calculus—the derivative and the integral. Each has a rich history and many practical applications.

Calculus is often described as the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

For example, a NASA scientist might need to know the initial velocity required for a rocket to escape Earth's gravitational field. Calculus is required to determine this escape velocity. An engineer might need to know the volume of a spherical object with a hole drilled through the center. The integral calculus is needed to compute this volume. Calculus is an important tool for economic predictions, such as the growth of the federal debt. Similarly, a biologist might want to calculate the growth rate of a population of bacteria, or a geologist might want to estimate the age of a fossil using carbon dating. In each of these cases, calculus is needed to solve the problem.

Although precalculus mathematics (geometry, algebra, and trigonometry) also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus. Precalculus mathematics is more static, whereas calculus is more dynamic. Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
- The curvature of a circle is constant and can be analyzed with precalculus mathematics. To analyze the variable curvature of a curve, you need calculus.
- The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

Our study of calculus will be presented in the same order as a university-level calculus course. The material is based on the 9th edition of the bestselling textbook *Calculus* by Ron Larson and Bruce H. Edwards (Brooks/Cole, 2010). However, any standard calculus textbok can be used for reference and support throughout the course.

As we progress through the course, most concepts will be introduced using illustrative examples. We will present all the important theoretical ideas and theorems but not dwell on their technical proofs. You will find that it is easy to understand and apply calculus to real-world problems without knowing these theoretical intracacies.

Graphing calculators and computers are playing an increasing role in the mathematics classroom. Without a doubt, graphing technology can enhance the understanding of calculus, so some instances where we use graphing calculators to verify and confirm calculus results have been included.

As we will see in this course, most of the applications of calculus can be modeled by the 2 major themes of calculus: the derivative and the integral. The essence of the derivative is the determination of the equation of the tangent line to a curve. On the other hand, the integral is best approached by determining the area bounded by the graph of a function.

We will begin our study of calculus with a course overview and a brief look at the tangent line problem. This interesting problem introduces the fundamental concept of a limit. Hence after a short, 2-lesson review of certain precalculus ideas, we will study limits. Then using limits, we will define the derivative and develop its properties. We will also present many applications of the derivative to science and engineering.

After this study of the derivative, we will turn to the integral, using another classic problem, the area problem, as an introduction. Despite the apparent differences between the derivative and the integral, we will see that they are intimately related by the surprising fundamental theorem of calculus. The remaining portion of the course will be devoted to integral calculations and applications. By the end of the course, we will have covered all the main topics of beginning calculus, including those covered in an Advanced Placement calculus AB course or a basic college calculus course.

Students are encouraged to use all course materials to their maximum benefit, including the video lessons, which they can review as many times as they wish; the individual lesson summaries and accompanying problems in the workbook; and the supporting materials in the back of the workbook, such as the solutions to all problems, glossary, list of formulas, list of theorems, trigonometry review sheet, and composite study sheet, which can be torn out and used for quick and easy reference.

Lesson One
A Preview of Calculus

Topics:

- Course overview.
- The tangent line problem.
- What makes calculus difficult?
- Course content and use.

Definitions and Formulas:

Note: Terms in bold correspond to entries in the Glossary or other appendixes.

- The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1},\ x_1 \neq x_2 .$$

- The **point-slope equation** of the line passing through the point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.

Summary:

In this introductory lesson, we talk about the content, structure, and use of the calculus course. We attempt to answer the question, what is calculus? One answer is that calculus is the mathematics of change. Another is that calculus is a field of mathematics with important applications in science, engineering, medicine, and business.

The principle example in this lesson is the classic tangent line problem: the calculation of the slope of the tangent line to a parabola at a specific point. This problem illustrates a core idea of the so-called differential calculus, a topic we study later.

Example 1: The Tangent Line to a Parabola

Find the slope and an equation of the tangent line to the parabola $y = x^2$ at the point $P = (2,4)$.

Solution:

Let $Q = (x, x^2)$, $x \neq 2$, be another point on the parabola. The slope of the line joining P and Q is as follows:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2,\ x \neq 2 .$$

Geometrically, as the point Q approaches P, the line joining P and Q approaches the tangent line at P. Algebraically, as x approaches 2, the slope of the line joining P and Q approaches the slope of the tangent line at P. Hence you see that the slope of the tangent line is $m = 2 + 2 = 4$. Symbolically, we represent this limit argument as follows:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2}(x + 2) = 4 .$$

The equation of the tangent line to the parabola at $(2,4)$ is $y-4=4(x-2)$, or $y=4x-4$.

The tangent line problem uses the concept of limits, a topic we will discuss in Lessons Four through Six.

Study Tip:

- You can use a graphing utility to verify that the tangent line intersects the parabola at a single point. To this end, graph $y=x^2$ and $y=4x-4$ in the same viewing window and zoom in near the point of tangency $(2,4)$.

Pitfalls:

- Calculus requires a good working knowledge of precalculus (algebra and trigonometry). We review precalculus in Lessons Two and Three. Furthermore, throughout the course we will point out places where algebra and trigonometry play a significant role. If your precalculus skills are not as dependable as you would like, you will want to have a good precalculus textbook handy to review and consult.
- Calculus also requires practice, so you will benefit from doing the problems at the end of each lesson. The worked-out solutions appear at the end of this workbook.

Problems:

1. Find the equation of the tangent line to the parabola $y=x^2$ at the point $(3,9)$.

2. Find the equation of the tangent line to the parabola $y=x^2$ at the point $(0,0)$.

3. Find the equation of the tangent line to the cubic polynomial $y=x^3$ at the point $(-1,-1)$.

Lesson Two
Review—Graphs, Models, and Functions

Topics:

- Sketch a graph of an equation by point plotting.
- Find the intercepts of a graph.
- Test a graph for symmetry with respect to an axis and the origin.
- Find the points of intersection of 2 graphs.
- Find the slope of a line passing through 2 points.
- Write the equation of a line with a given point and slope.
- Write equations of lines that are parallel or perpendicular to a given line.
- Use function notation to represent and evaluate a function.
- Find the domain and range of a function.

Definitions:

- The **intercepts** of a graph are the points where the graph intersects the x- or y-axis.
- A graph is **symmetric with respect to the y-axis** if whenever (x, y) is a point on the graph, $(-x, y)$ is also a point on the graph.
- A graph is **symmetric with respect to the x-axis** if whenever (x, y) is a point on the graph, $(x, -y)$ is also a point on the graph.
- A graph is **symmetric with respect to the origin** if whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph.
- A **point of intersection** of the graphs of 2 equations is a point that satisfies both equations.
- The delta notation (Δx) is used to describe the difference between 2 values: $\Delta x = x_2 - x_1$.
- The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2.$$

- Given 2 sets A and B, a **function** f is a correspondence that assigns to each number x in A exactly 1 number y in B. The set A is the **domain** of the function. The number y is the **image** of x under f and is denoted by $f(x)$. The **range** of f is the subset of B consisting of all the images.

Formulas:

- **Point-slope equation** of a line: $y - y_1 = m(x - x_1)$.
- **Slope-intercept equation** of a line: $y = mx + b$.

Summary:

This is the first of 2 lessons devoted to reviewing key concepts from precalculus. We review how to graph equations and analyze their symmetry. We look at the intercepts of a graph and how to determine where 2 graphs intersect each other. We then review the concept of slope of a line and look at various equations used to describe lines. In particular, we look at parallel and perpendicular lines. Finally, we begin the discussion of functions, recalling their definition and some important examples.

Example 1: Intercepts

Find the x- and y-intercepts of the graph of $y = x^3 - 4x$.

Solution:

To determine the x-intercepts, let $y = 0$ and solve for x:

$$0 = x^3 - 4x = x(x-2)(x+2).$$

Hence, the x-intercepts are $(0,0)$, $(2,0)$, and $(-2,0)$. To determine the y-intercepts, let $x = 0$ and solve for y. In this case, you obtain $(0,0)$.

Example 2: Points of Intersection

Find the points of intersection of the graphs of the equations $x^2 - y = 3$ and $x - y = 1$.

Solution:

Solve for y in each equation and then set them equal to each other:

$$y = x^2 - 3,\ y = x - 1 \text{ implies } x^2 - 3 = x - 1.$$

Solving this equation, you obtain $x^2 - x - 2 = (x-2)(x+1) = 0$. Hence $x = 2$ and $x = -1$, so the points of intersection are $(2,1)$ and $(-1,-2)$.

Example 3: Perpendicular Lines

Find the equation of the line passing through the point $(2,-1)$ and perpendicular to the line $3y - 2x + 5 = 0$.

Solution:

The given line can be written in slope-intercept form: $y = \frac{2}{3}x - \frac{5}{3}$. Because this line has slope $\frac{2}{3}$, the perpendicular line will have slope $\frac{-3}{2}$. Using the point-slope formula, you obtain $y - (-1) = \frac{-3}{2}(x-2)$, or $y = \frac{-3}{2}x + 2$.

Study Tips:

- You might need to plot many points to obtain a good graph of an equation.
- Use a graphing utility to verify your answers. For instance, in Example 2 above, try graphing the 2 equations on the same screen to visualize their points of intersection. Note that the Advanced Placement calculus examination requires a graphing utility.
- Horizontal lines have slope 0; their equations are of the form $y = b$, a constant.
- Slope is not defined for vertical lines; their equations are of the form $x = c$, a constant.
- Parallel lines have equal slopes.
- The slopes of perpendicular lines are negative reciprocals of each other.
- A vertical line can intersect the graph of a function at most once. This is called the **vertical line test**.

Pitfalls:

- In the formula for slope, make sure that $x_1 \neq x_2$. In other words, slope is not defined for vertical lines.
- In the formula for slope, the order of subtraction is important.
- On a graphing utility, you need to use a square setting for perpendicular lines to actually appear perpendicular.

Problems:

1. Sketch the graph of the equation $y = 4 - x^2$ by point plotting.

2. Find the intercepts of the graph of the equation $x\sqrt{16 - x^2}$.

3. Test the equation $y = \dfrac{x^2}{x^2 + 1}$ for symmetry with respect to each axis and the origin.

4. Find the points of intersection of the graphs of $x^2 + y = 6$ and $x + y = 4$. Verify your answer with a graphing utility.

5. Find the slope of the line passing through the points $(3, -4)$ and $(5, 2)$.

6. Determine an equation of the line that passes through the points $(2, 1)$ and $(0, -3)$.

7. Sketch the graphs of the equations $x + y = 1$, $y = -3$, and $x = 4$.

8. Find an equation of the line that passes through the point $(2, 1)$ and is perpendicular to the line $5x - 3y = 0$.

9. Determine the domain and range of the square root function $f(x) = \sqrt{x}$. Sketch its graph.

10. Determine the domain and range of the function $f(x) = x^3$. Sketch its graph.

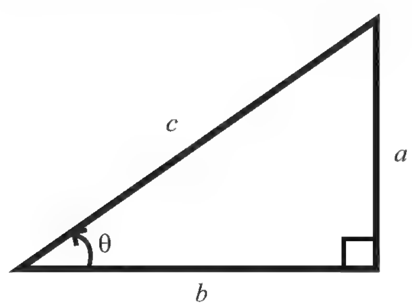
Lesson Three
Review—Functions and Trigonometry

Topics:

- Examples of functions.
- One-to-one functions.
- Even and odd functions.
- Radian and degree measure of angles.
- Triangle definition of the trigonometric functions.
- Unit-circle definition of the trigonometric functions.
- Trigonometric identities.
- Graphs of the trigonometric functions.
- Trigonometric equations.

Definitions:

- The **absolute value function** is defined as follows:
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$
- A function from X to Y is **one-to-one** if to each y-value in the range, there corresponds exactly 1 x-value in the domain.
- A function f is **even** if $f(-x) = f(x)$. A function f is **odd** if $f(-x) = -f(x)$.
- The **right-triangle definition** of the trigonometric functions uses the right triangle below.



$$\sin\theta = \frac{a}{c}$$
$$\cos\theta = \frac{b}{c}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a}{b}$$

- Let (x, y) be a point on the unit circle $x^2 + y^2 = 1$. The **unit-circle definitions** of the trigonometric functions are as follows: $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$, $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y}$.

Formulas:

- $360° = 2\pi$.
- $\sin^2\theta + \cos^2\theta = 1$.
- $\cos(-x) = \cos x$; $\sin(-x) = -\sin x$.

Summary:

This is the second lesson reviewing precalculus. After looking at some examples of functions, we recall one-to-one functions and talk about even and odd functions. Then we review the trigonometric functions, using both the right-triangle approach and the unit-circle definition. We also recall some basic identities and show how to solve trigonometric equations.

Example 1: The Absolute Value Function

The domain of the absolute value function is all real numbers, whereas the range is all nonnegative real numbers. It is an even function because $f(-x)=|-x|=|x|=f(x)$. The absolute value function is not one-to-one.

Example 2: Solving a Trigonometric Equation

Find all values of θ such that $\sin\theta=\frac{1}{2}$.

Solution:

Begin by drawing a unit circle and indicating where $y=\frac{1}{2}$. The corresponding angles are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Because you can add multiples of 2π to these angles, the final answer is $\theta=\frac{\pi}{6}+2k\pi, \frac{5\pi}{6}+2k\pi$, where k is an integer. Observe that if the question had asked for the values of θ in the interval $[0,2\pi]$, then you would have had only 2 solutions.

Study Tips:

- You can use the horizontal line test for one-to-one functions: If a horizontal line intersects the graph of a function at more than 1 point, then the function is not one-to-one. For example, $y=x^2$ is not one-to-one.
- A function must be one-to-one to have an inverse.
- Even functions are symmetric with respect to the y-axis, whereas odd functions are symmetric with respect to the origin. For example, the cosine function is even, and the sine function is odd.
- Be sure you can calculate the trigonometric functions for common angles, such as $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \pi, \ldots$.
- The fundamental identity $\sin^2 x+\cos^2 x=1$ is used throughout this course. Other identities will be indicated throughout the course, and a list can be found in the Review of Trigonometry section of your workbook.
- You should memorize the graphs of the 6 trigonometric functions. These graphs are in the Review of Trigonometry section of the workbook. Note especially that $|\sin x|\le 1$ and $|\cos x|\le 1$.
- If you feel you will require more review of precalculus topics, please consult a textbook on algebra or trigonometry.

Pitfalls:

- In calculus we use radian measure. A common error is to have your calculator set to degree mode. If a problem is stated in degree measure, you must convert to radians.
- Remember that you cannot divide by 0 in mathematics. For example, the domain of the function $f(x) = \dfrac{1}{x-1}$ is all real values except $x = 1$.
- The trigonometric functions are not one-to-one. You must restrict their domains to define their inverses.
- On Advanced Placement examinations, it is not necessary to fully simplify answers. In any case, you should consult your teacher about the need to simplify.

Problems:

1. Find the domain and range of the function $f(x) = x^2 - 5$.

2. Find the domain and range of the function $f(x) = -\sqrt{x+3}$.

3. Find the domain and range of the function $f(x) = \cot x$.

4. Evaluate the function

$$f(x) = \begin{cases} |x| + 1, x < 1 \\ -x + 1, x \geq 1 \end{cases}$$

at the points $x = -3,\ 1, 3,$ and $b^2 + 1$.

5. Sketch the graph of the function $f(x) = \dfrac{1}{3}x^3 + 3$ and find its domain and range.

6. Determine whether the function $f(x) = \sqrt[3]{x}$ is even or odd.

7. Determine whether the function $f(x) = x \cos x$ is even or odd.

8. Find the values of the 6 trigonometric functions corresponding to the angle $\dfrac{\pi}{6}$.

9. Use the identity $\sin^2 x + \cos^2 x = 1$ to derive the identity $\tan^2 x + 1 = \sec^2 x$.

10. Find all solutions to the trigonometric equation $2\sin^2 x - \sin x - 1 = 0$ on the interval $[0, 2\pi)$.

11. Find all solutions to the trigonometric equation $\tan x = 0$.

Lesson Four
Finding Limits

Topics:

- Informal definition of a limit.
- Finding limits graphically, numerically, and algebraically.
- Limits that fail to exist.
- Properties of limits.
- Two special trigonometric limits.

Definition:

- If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the **limit** of $f(x)$ as x approaches c is L. We write $\lim_{x \to c} f(x) = L$.

Formulas and Properties of Limits:

- $\lim_{x \to c} b = b, \lim_{x \to c} x = c, \lim_{x \to c} x^n = c^n$.
- If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$,

  1. $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$.

  2. $\lim_{x \to c} [bf(x)] = bL$.

  3. $\lim_{x \to c} [f(x)g(x)] = LK$.

  4. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \ K \neq 0$.

Summary:

The concept of a limit plays a fundamental role in the development of calculus. In this lesson, we define limits and show how to evaluate them. We also discuss properties of limits and look at different ways that a limit can fail to exist.

Example 1: Determining Limits

Find the limit: $\lim_{x \to 0} \dfrac{x}{\sqrt{x+1} - 1}$.

Solution:

Notice that you cannot just plug in the number 0 because that would yield the meaningless expression $\frac{0}{0}$.

You can approach this limit 3 ways. The first is to build a table of values for the function $f(x)=\frac{x}{\sqrt{x+1}-1}$ near $x=0$. The second is to graph the function and observe its behavior near $x=0$. In both cases, you will see that the function seems to approach 2 as a limit. The third method is to use an algebraic approach, which involves rewriting the limit as follows:

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}&=\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\\&=\lim_{x\to 0}\frac{x\left(\sqrt{x+1}+1\right)}{(x+1)-1}\\&=\lim_{x\to 0}\frac{x\left(\sqrt{x+1}+1\right)}{x}\\&=\lim_{x\to 0}\left(\sqrt{x+1}+1\right)=1+1=2.\end{aligned}$$

Notice that the original function, $f(x)=\frac{x}{\sqrt{x+1}-1}$, and $g(x)=\sqrt{x+1}+1$ agree at all points except $x=0$. Hence they have the same limit as x approaches 0.

Example 2: A Limit That Fails to Exist

For the function $f(x)=\frac{|x|}{x}$, evaluate the limit $\lim_{x\to 0}\frac{|x|}{x}$.

Solution:

As x approaches 0 from the right, we have $f(x)=\frac{|x|}{x}=\frac{x}{x}=1$. However, as x approaches 0 from the left, we have $f(x)=\frac{|x|}{x}=-\frac{x}{x}=-1$. Since the function f is not approaching a single real number L, the limit does not exist.

Example 3: Oscillating Behavior

For the function $f(x)=\sin\frac{1}{x}$, evaluate the limit $\lim_{x\to 0}\sin\frac{1}{x}$.

Solution:

If you graph this function near 0, you will see that the limit does not exist. You can prove this analytically by exhibiting a sequence of x-values that approach 0 for which the function is equal to 1 and another sequence of values that approach 0 for which the function is equal to -1. To this end, observe that $f(x)=1$ for $x=\frac{2}{\pi},\frac{2}{5\pi},\frac{2}{9\pi},\ldots,$ whereas $f(x)=-1$ for $x=\frac{2}{3\pi},\frac{2}{7\pi},\frac{2}{11\pi},\ldots$. Hence the limit does not exist at 0. The function oscillates infinitely often between -1 and 1.

A Useful Strategy for Finding Limits:

- If possible, use direct substitution. For example, $\lim_{x\to\pi}\cos x=\cos(\pi)=-1$.
- Try to find a function that agrees with the original function except at 1 point. The functions' limits will be the same.
- Verify your answer by a table of values or graph.

Special Trigonometric Limits:

The important limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ can be confirmed using a graphical or numerical analysis. The actual proof uses a geometric argument and the so-called squeeze theorem. You can find this proof in any calculus textbook. As a consequence of this limit, you see that for values of x near 0, $\sin x\approx x$. Using this limit, you can prove the related limit $\lim_{x\to 0}\frac{1-\cos x}{x}=0$.

Study Tips:

- The informal definition of limit given here is sufficient for our course. The formal definition is much more complicated and can be found in the Glossary or in any calculus text.
- The most precise mathematical way to evaluate a limit is the algebraic approach. You can use a table of values or a graphing utility to verify and confirm your algebraic calculations.

Pitfalls:

- When evaluating a limit as x approaches c, the value of the function at c does not matter. The function might not even be defined at c but could still have a limit.
- The graphical and numerical approaches to limits are not really proofs, but they are useful in verifying your analytic work.

Problems:

1. Find the limit: $\lim_{x\to 3}\frac{x^2-9}{x-3}$. Verify your answer with a graphing utility.

2. Find the limit: $\lim_{x\to\pi}\sin(2x)$.

3. Find the limit: $\lim_{x\to 4}\sqrt[3]{x+4}$.

4. Find the limit: $\lim_{x\to -3}\frac{x^2+x-6}{x+3}$. Verify your answer with a table of values.

5. Find the limit: $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$.

6. Find the limit: $\lim\limits_{x \to 4} \dfrac{\sqrt{x+5}-3}{x-4}$.

7. Discuss the existence of the following limit: $\lim\limits_{x \to -4} \dfrac{x+4}{|x+4|}$.

8. Discuss the existence of the following limit: $\lim\limits_{x \to 2} \dfrac{3}{x-2}$.

9. Find the following limit: $\lim\limits_{x \to 0} \dfrac{\sin 5x}{x}$.

10. Discuss the existence of $\lim\limits_{x \to 3} f(x)$, where

$$f(x) = \begin{cases} 5, x \neq 3 \\ 1, x = 3 \end{cases}.$$

Lesson Five
An Introduction to Continuity

Topics:

- Introduction to continuity.
- Definition of continuity and examples.
- Testing for continuity.
- The intermediate value theorem and applications.

Definitions and Theorems:

- A function f is **continuous** at c if the following 3 conditions are met:
  1. $f(c)$ is defined.
  2. $\lim_{x \to c} f(x)$ exists.
  3. $\lim_{x \to c} f(x) = f(c)$.
- The **limit from the right** means that x approaches c from values greater than c. The notation is $\lim_{x \to c^+} f(x) = L$. Similarly, the **limit from the left** means that x approaches c from values less than c, notated $\lim_{x \to c^-} f(x) = L$. These are called **one-sided limits**.
- The **greatest integer function** is defined $[\![x]\!] =$ the greatest integer n such that $n \le x$.
- The **intermediate value theorem** says that if f is continuous on the closed interval $[a,b]$, where $f(a) \ne f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least 1 number c in $[a,b]$ such that $f(c) = k$.

Formulas and Properties of Continuity:

- If f and g are continuous at c, then so are $f+g$, $f-g$, fg, and $\dfrac{f}{g}$ (where $g \ne 0$).
- The composition of continuous functions is continuous.

Summary:

Informally, we say that the function f is continuous at c if there is no interruption in the graph of f at c. That is, the graph is unbroken at c, and there are no holes, jumps, or gaps. The function $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{x-2} = x+2$, where $x \ne 2$, is not continuous at 2 because it is not defined there. The graph of this function is a line with a hole at the point $(2,4)$. However, this is a removable discontinuity because we can define that $f(2) = 4$, and the resulting function is now continuous everywhere. On the other hand, the greatest integer function is not continuous at all integer values, and these discontinuities are nonremovable.

Example 1: A Removable Discontinuity

Discuss the continuity of the function $f(x) = \dfrac{\sin x}{x}$.

Solution:

The function is continuous everywhere except 0, where it is not defined. By recalling the limit $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$, you see that this discontinuity is removable by defining that $f(0) = 1$.

Example 2: An Application of the Intermediate Value Theorem

Show that the polynomial $f(x) = x^3 + 2x - 1$ has a zero (root) in the interval $[0, 1]$.

Solution:

Notice that the function is continuous on the closed interval [0, 1]. Since $f(0) = -1$ and $f(1) = 2$, the intermediate value theorem tells us that there exists a number c in the interval $[0, 1]$ such that $f(c) = 0$. In fact, by using a graphing utility you can show that $c \approx 0.4534$.

Study Tips:

- Note that there are 3 parts to the definition of continuity at a point: The function must be defined at the point, the limit must exist at the point, and the limit must equal the function value at the point.
- The greatest integer function rounds down a given real number to an integer; it can be found on most graphing utilities.
- One-sided limits are useful when discussing continuity on closed intervals. For example, the semicircle given by $f(x) = \sqrt{4 - x^2}$ is continuous on the closed interval $[-2, 2]$ because it is continuous on the open interval $(-2, 2)$, and the 1-sided limits satisfy $\lim_{x \to -2^+} f(x) = \lim_{x \to 2^-} f(x) = 0$.

Pitfalls:

- Remember that a function can never be continuous at a point that is not in its domain. For example, the function $f(x) = \tan x$ is continuous for all x except $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$.
- The intermediate value theorem requires that the function be continuous on a closed interval. It is an existence theorem only.

Problems:

1. Find the following limit: $\lim_{x \to 0^+} \dfrac{|x|}{x}$.

2. Find the x-values, if any, at which the function $f(x) = \dfrac{6}{x}$ is not continuous. Which of the discontinuities are removable?

3. Find the x-values, if any, at which the function $f(x) = x^2 - 9$ is not continuous. Which of the discontinuities are removable?

4. Find the x-values, if any, at which the function $f(x) = \dfrac{x-6}{x^2 - 36}$ is not continuous. Which of the discontinuities are removable?

5. Discuss the continuity of the function $f(x) = \dfrac{x^2 - 1}{x + 1}$.

6. Discuss the continuity of the function $f(x) = \sqrt{49 - x^2}$.

7. Explain why the function $f(x) = x^3 + 5x - 3$ has a zero in the interval $[0,1]$.

8. Explain why the function $f(x) = x^2 - 2 - \cos x$ has a zero in the interval $[0, \pi]$.

9. True or false: The function $f(x) = \dfrac{|x-1|}{x-1}$ is continuous on $(-\infty, \infty)$.

10. Discuss the continuity of the function

$$f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases}.$$

Lesson Six
Infinite Limits and Limits at Infinity

Topics:

- Introduction to infinite limits.
- Vertical asymptotes.
- Introduction to limits at infinity.
- Horizontal asymptotes.
- Limits at infinity for rational functions.

Definitions and Theorems:

- $\lim\limits_{x \to c} f(x) = \infty$ means that $f(x)$ increases without bound as x approaches c.
- If $f(x)$ approaches infinity (or negative infinity) as x approaches c, from the right or left, then the vertical line $x = c$ is a **vertical asymptote** of the graph of f.
- The line $y = L$ is a **horizontal asymptote** of the graph of f if $\lim\limits_{x \to \infty} f(x) = L$ or $\lim\limits_{x \to -\infty} f(x) = L$.
- The **limit from the right** means that x approaches c from values greater than c. The notation is $\lim\limits_{x \to c^+} f(x) = L$. Similarly, the **limit from the left** means that x approaches c from values less than c, notated $\lim\limits_{x \to c^-} f(x) = L$.

Summary:

We use infinite limits to describe the behavior of functions that increase (or decrease) without bound. If the graph of a function becomes arbitrarily close to a vertical line $x = c$, we say that this line is a vertical asymptote of the graph. For instance, the function $f(x) = \tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

Example 1: Determining Infinite Limits

Determine the 1-side limits $\lim\limits_{x \to 2^+} \frac{3}{x-2}$ and $\lim\limits_{x \to 2^-} \frac{3}{x-2}$.

Solution:

As x approaches 2 from the right, the values of the function $f(x) = \frac{3}{x-2}$ increase without bound. For example, if $x = 2.001$, $f(2.001) = 3000$. We express this by writing $\lim\limits_{x \to 2^+} \frac{3}{x-2} = \infty$. Similarly, as x approaches 2 from the left, the values of the function decrease without bound, and we write $\lim\limits_{x \to 2^-} \frac{3}{x-2} = -\infty$. In both cases the limits fail to exist, but the symbol ∞ denotes the unbounded behavior of the function.

Example 2: Finding Vertical Asymptotes

Determine the vertical asymptotes of the function $f(x)=\dfrac{x^2+2x-8}{x^2-4}$.

Solution:

Factor the numerator and denominator as follows:

$$f(x)=\frac{x^2+2x-8}{x^2-4}=\frac{(x-2)(x+4)}{(x-2)(x+2)}=\frac{x+4}{x+2},\ x\neq 2\,.$$

From this expression, we see that $x=-2$ is a vertical asymptote, whereas $x=2$ is not. In fact, there is a hole at $x=2$. Notice that you cannot just assume that the vertical asymptotes occur at zeros of the denominator. Some of these might be zeros of the numerator.

In the next example, we use limits at infinity to describe the behavior of functions as x tends to infinity or negative infinity.

Example 3: Finding a Limit at Infinity

Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{2x^2+5}{3x^2+1}$.

Solution:

Multiply the numerator and denominator by $\dfrac{1}{x^2}$:

$$\lim_{x\to\infty}\frac{2x^2+5}{3x^2+1}=\lim_{x\to\infty}\frac{(2x^2+5)\left(\frac{1}{x^2}\right)}{(3x^2+1)\left(\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{2+\frac{5}{x^2}}{3+\frac{1}{x^2}}=\frac{2}{3}\,.$$

Hence the line $y=\dfrac{2}{3}$ is a horizontal asymptote. Notice that if the numerator were changed to $2x+5$, then the previous argument would show that the limit is 0. And if the numerator were $2x^3+5$, then the limit would be infinity.

Study Tips:

- How do you determine limits at infinity for rational functions (quotients of polynomials)?
  1. If the degree of the numerator is less than the degree of the denominator, the limit at infinity is 0.
  2. If the degree of the numerator is equal to the degree of the denominator, the limit at infinity is the ratio of the leading coefficients.
  3. If the degree of the numerator is greater than the degree of the denominator, the limit at infinity does not exist (it is infinity or negative infinity).
- It is possible for a graph to have 0, 1, or even 2 horizontal asymptotes. For instance, the function $f(x)=\dfrac{3x}{\sqrt{x^2+2}}$ has the horizontal asymptote $y=3$ to the right and $y=-3$ to the left.
- A graph can have any number of vertical asymptotes. For example, the function $f(x)=\tan x$ has an infinite number of vertical asympotores.

Pitfalls:

- Keep in mind that infinity is not a number. The equal sign in the limit $\lim_{x \to c} f(x) = \infty$ does not mean that the limit exists. On the contrary, it tells you how the limit fails to exist by denoting the unbounded behavior of $f(x)$ as x approaches c.
- To find vertical asymptotes, you cannot just use the zeros of the denominator. Some factors of the denominator might cancel with like factors in the numerator.
- You should use a graphing utility to confirm the existence of vertical and horizontal asymptotes. But be aware that the graph might give a misleading result if the viewing screen is not chosen carefully.

Problems:

1. Determine whether $f(x) = \dfrac{1}{x-4}$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

2. Determine whether $f(x) = \dfrac{-1}{(x-4)^2}$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

3. Find the vertical asymptotes, if any, of the function $f(x) = \dfrac{x^2}{x^2-4}$.

4. Find the vertical asymptotes, if any, of the function $f(x) = \dfrac{x^2-1}{x+1}$.

5. Find the limit: $\lim\limits_{x \to -3^-} \dfrac{x+3}{x^2+x-6}$.

6. Find the limit: $\lim\limits_{x \to \infty} \dfrac{x^2+2}{x^3-1}$.

7. Find the limit: $\lim\limits_{x \to \infty} \dfrac{x^2+2}{x^2-1}$.

8. Find the limit: $\lim\limits_{x \to -\infty} \left(\dfrac{5}{x} - \dfrac{x}{3} \right)$.

9. Find the limit: $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$.

10. Use a graphing utility to identify the horizontal asymptotes of the function $f(x) = \dfrac{3x-2}{\sqrt{2x^2+1}}$.

Lesson Seven
The Derivative and the Tangent Line Problem

Topics:

- The tangent line problem revisited.
- The definition of slope.
- The definition and notation of the derivative.
- Using the definition to calculate derivatives.
- Examples of functions that are not differentiable at a point.
- The relationship between continuity and differentiability.

Definitions and Theorems:

- The **slope of the graph of the function** f at the point $(c, f(c))$ is defined as the following limit, if it exists: $m = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.
- By letting $x = c + \Delta x,\ x - c = \Delta x,$ you obtain the equivalent definition of **slope**:
$$m = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$
- The **derivative** of f at x is given by the following limit, if it exists:
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$
- **Differentiability implies continuity theorem**: If f is differentiable at $x = c$, then f is continuous at $x = c$. The converse is not true (see Examples 2 and 3 below).

Notations:

- There are many notations for the derivative of $y = f(x)$: $f'(x),\ \frac{dy}{dx},\ y',\ \frac{d}{dx}[f(x)],\ D[y]$.

Summary:

The definition of the derivative is a generalization of the slope of the tangent line to a curve. It can be difficult to use this definition, but sometimes it is necessary. Later on, we will develop convenient formulas that will permit us to calculate the derivative of almost any function we encounter.

Example 1: Finding the Derivative by the Limit Process

Find the derivative of $f(x) = x^3$.

Solution:

We use the limit definition as follows:

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{(x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3)-x^3}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\Delta x(3x^2+3x(\Delta x)+(\Delta x)^2)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left(3x^2+3x(\Delta x)+(\Delta x)^2\right)=3x^2.\end{aligned}$$

That is, $\frac{d}{dx}\left[x^3\right]=3x^2.$

Example 2: A Graph with a Sharp Turn

The function $f(x)=|x-2|$ is continuous for all real numbers. Is it differentiable at $c=2$?

Solution:

We will use the tangent line formula at $c=2$:

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to 2}\frac{|x-2|-0}{x-2}=\lim_{x\to 2}\frac{|x-2|}{x-2}.$$

By analyzing the 1-sided limits near 2, you see that this limit does not exist. Hence the function is not differentiable at 2. Geometrically, the graph has a sharp corner at 2 and is not smooth there.

Example 3: A Graph with a Vertical Tangent Line

Is the function $f(x)=x^{\frac{1}{3}}$ differentiable at the point (0, 0)?

Solution:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^{\frac{1}{3}}-0}{x}=\lim_{x\to 0}\frac{1}{x^{\frac{2}{3}}}=\infty$$

Although the function is continuous at (0, 0), it is not differentiable there. In fact, the graph of f has a vertical tangent at this point. Recall in general that differentiability implies continuity, but the converse is false.

Study Tips:

- The symbol Δx denotes a (small) change in x. Some textbooks use h instead of Δx.
- If the limit in the definition of slope is infinity (or negative infinity), we say that the graph has a vertical tangent at the point of tangency.

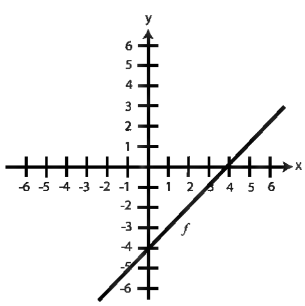
- Keep in mind that the definitions of slope and the derivative are based on the difference quotient for slope:
$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$
- Other letters can be used for the independent variable x and the dependent variable y. For instance, we will use $s(t)$ to model the position of an object with respect to time, t.

Pitfalls:

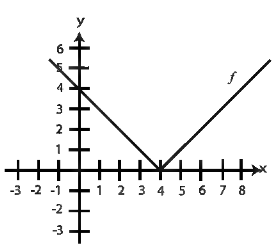
- Remember that continuity at a point does not imply differentiability there. A good example is the absolute value function, which is not differentiable at 0.
- When using the limit definition of derivative, you will not be able to substitute 0 for Δx, since you cannot divide by 0 in mathematics. Hence you will need to use algebraic skills to simplify the difference quotient, as indicated in Example 1 above.

Problems:

1. Find the slope of the tangent line to the graph of $f(x) = 3 - 5x$ at the point $(-1, 8)$.
2. Find the slope of the tangent line to the graph of $f(x) = x^2$ at the point $(3, 9)$.
3. Find the derivative of $f(x) = 7$ by the limit process.
4. Find the derivative of $f(x) = 3x + 2$ by the limit process.
5. Find the derivative of $f(x) = \dfrac{1}{x^2}$ by the limit process.
6. Find the equation of the tangent line to the graph $f(x) = \sqrt{x}$ at the point $(1,1)$. Use a graphing utility to graph the function and tangent line in the same viewing window.
7. Determine whether or not the function $f(x) = |x + 7|$ is differentiable everywhere.
8. Determine whether the function $f(x) = (x - 6)^{\frac{2}{3}}$ is differentiable everywhere.
9. Given the graph of f, sketch the graph of f'.



10. Given the graph of f, sketch the graph of f'.



Lesson Eight
Basic Differentiation Rules

Topics:

- The derivative of a constant function.
- The power rule.
- The constant multiple rule.
- Sum and difference rules.
- The derivatives of the sine and cosine functions.
- Rates of change and motion.

Definitions and Theorems:

- The derivative of a constant function is 0.
- The power rule says that $\frac{d}{dx}\left[x^n\right]=nx^{n-1}$.
- The constant multiple rule says that $\frac{d}{dx}\left[cf(x)\right]=cf'(x)$, where c is a constant.
- Sum and difference rules say that differentiation is linear: $\frac{d}{dx}\left[f(x)\pm g(x)\right]=f'(x)\pm g'(x)$.
- Derivatives of the sine and cosine functions: $\frac{d}{dx}\left[\sin x\right]=\cos x$; $\frac{d}{dx}\left[\cos x\right]=-\sin x$.

Formulas and Properties:

The position s of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation $s(t)=\frac{1}{2}gt^2+v_0t+s_0$, where t is time, s_0 is the initial height, v_0 is the initial velocity, and g is the acceleration due to gravity. On Earth, $g\approx -32$ ft/sec$^2\approx -9.8$ m/sec^2.

Summary:

We begin to develop convenient rules for evaluating derivatives. You can see that the derivative of a constant is 0 because the graph is a horizontal line. The power rule says, for example, that $\frac{d}{dx}\left[x^3\right]=3x^2$, as we have seen previously using the limit definition.

Example 1: Finding the Derivative

1. $\frac{d}{dx}\left[x^{\frac{2}{3}}\right]=\frac{2}{3}\left(x^{\frac{2}{3}-1}\right)=\frac{2}{3x^{\frac{1}{3}}}$.
2. $\frac{d}{dx}\left[\frac{1}{x^2}\right]=\frac{d}{dx}\left[x^{-2}\right]=-2x^{-3}=\frac{-2}{x^3}$.

In the next examples, we see how the rules developed in this lesson permit us to evaluate many derivatives.

Example 2: Finding the Derivative Using Sum and Difference Rules

1. $\dfrac{d}{dx}\left[x^3 - 4x + 5\right] = 3x^2 - 4$.
2. $\dfrac{d}{dt}\left[3t^5 - 14\pi\right] = 15t^4$.

Example 3: Trigonometric Derivatives

1. $y = 3\sin x + \pi$ implies $y' = 3\cos x$.
2. $y = 5x^2 + \frac{1}{3}\cos x$ implies $y' = 10x - \frac{1}{3}\sin x$.

In the next example we use $s(t)$ as the position of a particle, and hence its velocity is the derivative, $v(t) = s'(t)$.

Example 4: Motion

At time $t = 0$, a diver jumps up from a platform diving board that is 32 feet above the water. The position of the diver is given by $s(t) = -16t^2 + 16t + 32$, where s is in feet and t is in seconds. When will the diver hit the water, and what is the velocity at impact?

Solution:

Set the position function equal to 0, and solve for t.

$$-16t^2 + 16t + 32 = 0$$
$$-16(t^2 - t - 2) = 0$$
$$16(t+1)(t-2) = 0$$

Hence, $t = 2$ seconds. Because $v(t) = s'(t) = -32t + 16$, $v(2) = -32(2) + 16 = -48$ feet per second is the velocity at impact.

Study Tips:

- You might have to simplify an expression to use the power rule, as illustrated in the second part of Example 1 above.
- Don't forget the slope interpretation of the derivative. Because the derivative of $f(x) = x^2$ is $f'(x) = 2x$, you see that the parabola has a negative slope for $x < 0$ and a positive slope for $x > 0$. Furthermore, the slope is 0 at $x = 0$.
- You still might need the original definition of the derivative for some problems. For example, you need the definition to determine that the derivative of $f(x) = x|x|$ at $x = 0$ is 0.
- You have probably observed that you need to use precalculus skills to solve many of the examples so far. Please feel free to review these skills as you need them.

Pitfalls:

- Remember that the derivative of cosine has a minus sign: $(\cos x)' = -\sin x$.
- The derivative of a product is not the product of the derivatives, nor is the derivative of a quotient the quotient of the derivatives.

Problems:

1. Find the derivative of $f(x) = \sqrt[5]{x}$.
2. Find the derivative of $s(t) = t^3 + 5t^2 - 3t + 8$.
3. Find the derivative of $f(x) = \dfrac{1}{x^5}$.
4. Find the derivative of $f(x) = \dfrac{\sqrt{x}}{x}$.
5. Find the derivative of $f(x) = 6\sqrt{x} + 5\cos x$.
6. Find the equation of the tangent line to the graph $f(x) = 3x^3 - 10$ at the point $(2,14)$. Use a graphing utility to graph the function and tangent line in the same viewing window.
7. Find the equation of the tangent line to the graph $f(\theta) = 4\sin\theta - \theta$ at the point $(0,0)$. Use a graphing utility to graph the function and tangent line in the same viewing window.
8. Find the equation of the tangent line to the graph $f(x) = (x^2 + 2)(x + 1)$ at the point $(1,6)$. Use a graphing utility to graph the function and tangent line in the same viewing window.
9. Determine the point(s) at which the graph of $f(x) = x^4 - 2x^2 + 3$ has a horizontal tangent line.
10. A silver dollar is dropped from a building that is 1362 feet tall. Determine the position and velocity functions for the coin. When does the coin hit the ground? Find the velocity of the coin at impact.

Lesson Nine
Product and Quotient Rules

Topics:

- The product rule.
- The quotient rule.
- Higher-order derivatives.
- An introduction to differential equations.

Definitions and Theorems:

- **The product rule:** The derivative of the product of 2 differential functions f and g is $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$.
- **The quotient rule**: The derivative of the quotient of 2 differential functions f and g is $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.
- A **differential equation** is an equation containing derivatives.

Formulas and Properties:

$$[\sin x]' = \cos x. \qquad [\cos x]' = -\sin x.$$
$$[\tan x]' = \sec^2 x. \qquad [\cot x]' = -\csc^2 x.$$
$$[\sec x]' = \sec x \tan x. \qquad [\csc x]' = -\csc x \cot x.$$

Summary:

The derivative of a product of 2 functions is not the product of their derivatives. Nor is the derivative of a quotient of 2 functions the quotient of their derivatives. The product rule is somewhat complicated, as illustrated in the following example.

Example 1: The Product Rule

$$\begin{aligned}\frac{d}{dx}\left[3x^2 \sin x\right] &= 3x^2 \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}\left[3x^2\right] \\ &= 3x^2 \cos x + (\sin x)6x \\ &= 3x(x\cos x + 2\sin x)\end{aligned}$$

Example 2: The Quotient Rule

$$\begin{aligned}\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] &= \frac{(x^2+1)\frac{d}{dx}[5x-2]-(5x-2)\frac{d}{dx}\left[x^2+1\right]}{(x^2+1)^2}\\ &= \frac{(x^2+1)5-(5x-2)(2x)}{(x^2+1)^2}\\ &= \frac{5x^2+5-10x^2+4x}{(x^2+1)^2}\\ &= \frac{-5x^2+4x+5}{(x^2+1)^2}\end{aligned}$$

In the next example, we look at higher-order derivatives.You obtain the second derivative of a function by taking the derivative of the first derivative. These higher-order derivatives are useful in applications. For example, velocity is the first derivative of the position function, and acceleration is the second derivative: $v(t)=s'(t)$, and $a(t)=s''(t)=v'(t)$.

Example 3: Higher-Order Derivatives

$$\begin{aligned}&f(x) = x^4\\ &f'(x) = 4x^3\\ &f''(x) = 12x^2\\ &f'''(x) = 24x\\ &f^{(4)}(x) = 24\\ &f^{(5)}(x) = 0\end{aligned}$$

Example 4: Differential Equations

Show that the function $y=\sin x$ satisfies the differential equation $y''+y=0$.

Solution:

We have $y'=\cos x$ and $y''=-\sin x$. Substituting these functions into the differential equation, we have $y''+y=-\sin x+\sin x=0$.

Study Tips:

- For the Advanced Placement examination, it is not necessary to simplify answers. But this might not be true in a college course.
- You don't always need to use the quotient rule for a quotient of 2 functions. For instance, you can easily differentiate the function $f(x)=\dfrac{x^3+\cos x}{6}$ by factoring out the 6 in the denominator.

Pitfalls:

- The derivative of the product of 2 functions is not the product of their derivatives. You can see this by letting $f(x) = x$ and $g(x) = x^2$. Similarly, the derivative of a quotient of 2 functions is not the quotient of their derivatives.
- When calculating derivatives, be aware that the answer you obtain might not look like the answer in the back of the textbook. Remember, there can be many equivalent forms for a derivative depending on how you simplify.

Problems:

1. Use the product rule to find the derivative of $f(x) = x^3 \cos x$.

2. Use the product rule to find the derivative of $f(x) = (x^2 + 3)(x^2 - 4x)$.

3. Use the quotient rule to find the derivative of $f(x) = \dfrac{x}{x^2 + 1}$.

4. Use the quotient rule to find the derivative of $f(x) = \dfrac{\sin x}{x^2}$.

5. Find the equation of the tangent line to the graph $f(x) = (x^3 + 4x - 1)(x - 2)$ at the point $(1, -4)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

6. Find the equation of the tangent line to the graph $f(x) = \dfrac{x-1}{x+1}$ at the point $\left(2, \dfrac{1}{3}\right)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

7. Find the derivative of $f(x) = \dfrac{10}{3x^3}$ without the quotient rule.

8. Find the derivative of the trigonometric function $f(x) = x^2 \tan x$.

9. Find the derivative of the trigonometric function $f(x) = \dfrac{\sec x}{x}$.

10. Find the second derivative of $f(x) = 4x^{\frac{3}{2}}$.

11. Find the second derivative of $f(x) = \dfrac{x}{x-1}$.

12. Verify that the function $y = \dfrac{1}{x}$, $x > 0$, satisfies the differential equation $x^3 y'' + 2x^2 y' = 0$.

Lesson Ten
The Chain Rule

Topics:

- The chain rule.
- The general power rule.
- Repeated chain rule.

Definitions and Theorems:

- **Chain rule:** If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x, and $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$, or equivalently, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Summary:

The chain rule tells how to find the derivative of a composition of 2 functions. Notice in the example below that $f(x) = \left(x^2+1\right)^3$ is the composition of the functions $y = f(u) = u^3$ and $u = g(x) = x^2+1$. Hence $\frac{dy}{du} = 3u^2 = 3\left(x^2+1\right)^2$, and $\frac{du}{dx} = 2x$.

Example 1: The Chain Rule

$$\frac{d}{dx}\left[\left(x^2+1\right)^3\right] = \frac{dy}{du}\frac{du}{dx} = 3\left(x^2+1\right)^2(2x) = 6x(x^2+1)^2$$

I like to say that the chain rule is like the ordinary rules of differentiation if you remember to multiply by the "inside derivative." In particular, the power rule $\left[x^n\right]' = nx^{n-1}$ becomes $\left[u^n\right]' = nu^{n-1}\frac{du}{dx}$.

Notice the inside derivative 3 in the next example.

Example 2: The General Power Rule

$$\frac{d}{dx}\left[\sqrt{3x+1}\right] = \frac{d}{dx}\left[\left(3x+1\right)^{\frac{1}{2}}\right] = \frac{1}{2}\left(3x+1\right)^{-\frac{1}{2}}(3) = \frac{3}{2\sqrt{3x+1}}$$

Sometimes you need to apply the chain rule more than once.

Example 3: Repeated Chain Rule

$$\begin{aligned}\frac{d}{dx}\left[\sin^3(4x)\right] &= \frac{d}{dx}\left[(\sin 4x)^3\right]\\ &= 3(\sin 4x)^2\frac{d}{dx}[\sin 4x]\\ &= 3(\sin 4x)^2\cos(4x)4\\ &= 12\sin^2 4x\cos 4x\end{aligned}$$

Study Tips:

- If you are asked to find the derivative of a function at a specific point, calculate the derivative and immediately plug in the point. Do not simplify ahead of time.
- You might not need to apply the chain rule for some problems. For instance, the derivative of $f(x)=\sin^2 x+\cos^2 x$ is 0 because $f(x)=1$, a constant.

Pitfall:

- On multiple-choice tests, the form of your answer is important. You might have the correct answer but need to simplify it to match one of the choices on the test.

Problems:

1. Use the chain rule to find the derivative of $f(x)=(4x-1)^3$.

2. Use the chain rule to find the derivative of $f(t)=\sqrt{5-t}$.

3. Use the chain rule to find the derivative of $f(x)=\cos 4x$.

4. Use the chain rule to find the derivative of $f(x)=5\tan 3x$.

5. Find the equation of the tangent line to the graph $f(x)=\left(9-x^2\right)^{\frac{2}{3}}$ at the point $(1,4)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

6. Find the equation of the tangent line to the graph $f(x)=26-\sec^3 4x$ at the point $(0,25)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

7. Find the derivative of $f(x)=\sin(\tan 2x)$.

8. Find the derivative of $f(x)=\left(\dfrac{3x-1}{x^2+3}\right)^2$.

9. Find the derivative of the function $f(x)=\dfrac{5}{x^3-2}$ at the point $\left(-2,-\dfrac{1}{2}\right)$.

10. Find the second derivative of $f(x)=\dfrac{4}{(x+2)^3}$.

11. Find the derivative of $f(x)=\tan^2 x-\sec^2 x$. What do you observe?

Lesson Eleven
Implicit Differentiation and Related Rates

Topics:

- Explicit and implicit functions.
- Implicit differentiation.
- Related rates.

Guidelines for Implicit Differentiation:

- Differentiate both sides of the equation with respect to x.
- Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation, and move all other terms to the right side of the equation.
- Factor $\frac{dy}{dx}$ out of the left side of the equation.
- Solve for $\frac{dy}{dx}$.

Summary:

Functions like $f(x)=\sqrt{25-x^2}$ are said to be explicitly defined, whereas equations like the circle $x^2+y^2=25$ do not define y as an explicit function of x. In fact, the circle equation above gives rises to 2 functions, the upper and lower semicircles $f(x)=\sqrt{25-x^2}$ and $g(x)=-\sqrt{25-x^2}$. You can find the derivative y' without actually solving for y, as indicated in Example 1 below. Just remember that y is considered to be a function of x.

Example 1: Implicit Differentiation

Find the slope of the tangent line to the graph of $x^2+y^2=25$ at the point $(3,4)$.

Solution:

Use implicit differentiation.

$$\begin{aligned} x^2+y^2&=25 \\ 2x+2yy'&=0 \\ 2yy'&=-2x \\ y'&=\frac{-x}{y} \end{aligned}$$

At the point $(3,4)$, $y'=\frac{-3}{4}$. You should verify this calculation by differentiating the function corresponding to the upper semicircle.

Example 2: Implicit Differentiation and the Sine Function

Find $\frac{dy}{dx}$ if $\sin y = x,\ \frac{-\pi}{2} < y < \frac{\pi}{2}$.

Solution:

We use implicit differentiation. Note that $\cos y$ is positive on the given interval and that we can replace $\sin y$ with x in the final step.

$$\sin y = x$$
$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Up to this point in the course, y has been a function of the independent variable x. Now we will look at related rates problems in which x and y are functions of a third variable, t. As t varies, so do x and y. In the next example, involving a circle, the variables r (radius) and A (area) are changing with respect to t (time).

Example 3: Related Rates

A pebble is dropped into a calm pond, causing circular ripples. The radius r is increasing at a rate of 1 foot per second. At what rate is the area A of the circle changing when the radius is 4 feet? When it is 8 feet?

Solution:

The 2 quantities that are changing are related by the equation for the area of a circle, $A = \pi r^2$. We differentiate both sides with respect to t and replace $\frac{dr}{dt}$ with 1.

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r$$

At $t = 4$, $\frac{dA}{dt} = 8\pi$, and at $t = 8$, $\frac{dA}{dt} = 16\pi$. Note that the rate of increase of area is increasing.

Study Tip:

- Related rates problems can be difficult for 2 reasons. First, they are word problems that often require careful reading. Second, related rates problems use formulas from precalculus: areas of circles, volumes of spheres, similar triangles, the pythagorean theorem, and so on. I suggest drawing a picture of the situation under consideration and labeling all the appropriate parts.

Pitfall:

- In applied calculus problems, you will see many different names for the variables. For example, the variable θ might represent an angle. Keep in mind that the underlying principles are the same.

Problems:

1. Find y' by implicit differentiation: $x^2 + y^2 = 9$.
2. Find y' by implicit differentiation: $x^3 - xy + y^2 = 7$.
3. Find y' by implicit differentiation: $y = \sin xy$.
4. Find y' by implicit differentiation: $\tan(x + y) = x$.
5. Find the slope of the tangent line to the graph $(4-x)y^2 = x^3$ at the point $(2,2)$.
6. Find an equation of the tangent line to the graph of $(x+2)^2 + (y-3)^2 = 37$ at the point $(4,4)$.
7. Find the second derivative y'' if $y^2 = x^3$.
8. Air is being pumped into a spherical balloon at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.
9. A 25-foot ladder is leaning against the wall of a house. The base of the ladder is being pulled away from the wall at a rate of 2 feet per second. How fast is the top of the ladder moving down the wall when the base is 7 feet from the wall?
10. A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

Lesson Twelve
Extrema on an Interval

Topics:

- Maximum and minimum values of functions.
- The extreme value theorem.
- Relative extrema.
- Critical numbers.

Definitions and Theorems:

- Let f be defined on an open interval I containing c. We know that $f(c)$ is the **minimum** of f on I if $f(c) \leq f(x)$ for all x in I. Similarly, $f(c)$ is the **maximum** of f on I if $f(c) \geq f(x)$ for all x in I. These are the **extreme values**, or **extrema**, of f. We sometimes say these are the **absolute minimum and maximum values** of f on I.
- The **extreme value theorem** states that if f is continuous on a closed interval $[a,b]$, then f has both a minimum and a maximum on the interval.
- If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** (or local maximum) of f.
- If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** (or local minimum) of f.
- Let f be defined at c. If $f'(c)=0$ or f is not defined at c, then c is a **critical number** of f.
- Theorem: If f has a relative minimum or relative maximum at $x=c$, then c is a critical number of f.

Guidelines for Finding Extrema on a Closed Interval:

Let f be continuous on the closed interval $[a,b]$.

1. Find the critical numbers of f on the open interval (a,b).
2. Evaluate f at each critical number.
3. Evaluate f at each endpoint.
4. Select the greatest and least of these values.

Summary:

We will use calculus to find the absolute maximum and minimum values of a given function. These functions might be models of real-life situations, and finding their extrema is called optimization. We will see that the maximum and minimum values of a continuous function defined on a closed interval will always occur either at an endpoint or a point where the derivative is 0 or undefined.

Example 1: Finding Extrema on a Closed Interval

Find the maximum and minimum values of the function $f(x)=x^2-2x$ on the interval $(0,4)$.

Solution:

The derivative $f'(x)=2x-2$ is 0 for $x=1$. We evaluate the function at this point and at the 2 endpoints.

$$f(1)=-1$$
$$f(0)=0$$
$$f(4)=8$$

The maximum value is 8, and the minimum value is -1. The points in the domain of the function where the derivative is 0 or does not exist are called the critical numbers. These numbers, together with the endpoints, are the candidates for absolute extrema.

Example 2:

Find the absolute maximum and minimum values of $f(x)=2x-3x^{\frac{2}{3}}$ on the interval $[-1,3]$.

Solution:

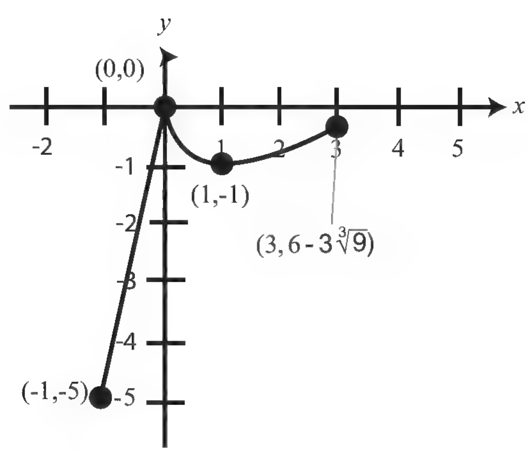
We calculate the derivative to find the critical numbers. Note the difficult factoring steps.

$$f(x)=2x-3x^{\frac{2}{3}}$$
$$f'(x)=2-2x^{\frac{-1}{3}}=2\left(1-\frac{1}{x^{\frac{1}{3}}}\right)=2\left(\frac{x^{\frac{1}{3}}-1}{x^{\frac{1}{3}}}\right)$$

The critical numbers are 1 (where the derivative is 0) and 0 (where the derivative is undefined). We now evaluate the function at these 2 points and the endpoints.

$$f(1)=-1$$
$$f(0)=0$$
$$f(-1)=-5$$
$$f(3)=6-3(3^{\frac{2}{3}})\approx-0.24$$

The absolute maximum is 0, and the absolute minimum is -5. Notice on the graph that there is a relative maximum at (0, 0) and a relative minimum at $(-1,-5)$. The graph is smooth at $(-1,-5)$ and has a sharp corner, or cusp, at (0, 0).



Study Tips:

- The extreme value theorem is an existence theorem. It guarantees the existence of the absolute extrema, but we need calculus to actually find these values.
- The critical numbers of a function yield all the candidates for relative extrema on an open interval.

Pitfalls:

- It is important to read an optimization problem carefully. Are they asking for the absolute maximum value, $f(c)$, or just the coordinate c where this value occurs?
- Critical numbers do not always yield a relative extremum. For example, 0 is a critical number of $f(x)=x^3$, but 0 is not a relative extremum.
- Critical numbers must be in the domain of the function. For example, 0 is not a critical number of the function $f(x)=\dfrac{1}{x}$ even though the derivative is not defined at 0. In fact, 0 is not in the domain of the function.

Problems:

1. Find the critical numbers of the function $f(x)=x^4-4x^2$.
2. Find the critical numbers of the function $h(x)=\sin^2 x+\cos x,\ 0<x<2\pi$.
3. Find the value of the derivative of the function $f(x)=\dfrac{x^2}{x^2+4}$ at its relative minimum $(0,0)$.
4. Find the value of the derivative of the function $f(x)=4-|x|$ at its relative maximum $(0,4)$.
5. Find the absolute extrema of the function $y=-x^2+3x-5,\ [-2,1]$.
6. Find the absolute extrema of the function $y=x^3-\dfrac{3}{2}x^2,\ [-1,2]$.
7. Find the absolute extrema of the function $f(t)=3-|t-3|,\ [-1,5]$.
8. Find the absolute extrema of the function $y=3x^{\frac{2}{3}}-2x,\ [-1,1]$.
9. Locate the absolute extrema of the function $f(x)=2x-3$ (if any exist) on the following intervals:
   a. $[0,2]$
   b. $[0,2)$
   c. $(0,2]$
   d. $(0,2)$
10. True or false: The maximum of a function that is continuous on a closed interval can occur at 2 different values in the interval.

Lesson Thirteen
Increasing and Decreasing Functions

Topics:

- Increasing and decreasing functions.
- The first derivative test.
- Rolle's theorem.
- The mean value theorem.

Definitions and Theorems:

- A function f is **increasing** on an interval I if for any 2 numbers a and b in the interval, $a<b$ implies $f(a)<f(b)$. Similarly, a function f is **decreasing** on an interval I if for any 2 numbers a and b in the interval, $a<b$ implies $f(a)>f(b)$.
- The **first derivative test**: Let c be a critical number of f. If f' changes from positive to negative at c, then f has a relative maximum at $(c,f(c))$. If f' changes from negative to positive at c, then f has a relative minimum at $(c,f(c))$.
- **Rolle's theorem**: Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$, then there is at least 1 number c in (a,b) such that $f'(c)=0$.
- **Mean value theorem**: Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Then there is at least 1 number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Test for Increasing and Decreasing Functions:

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

1. If $f'(x)>0$ for all x in (a,b), then f is increasing on $[a,b]$.
2. If $f'(x)<0$ for all x in (a,b), then f is decreasing on $[a,b]$.
3. If $f'(x)=0$ for all x in (a,b), then f is constant on $[a,b]$.

Guidelines for Finding Intervals on Which a Function Is Increasing or Decreasing:

Let f be continuous on (a,b).

1. Locate the critical numbers of f.
2. Use these numbers to determine test intervals.
3. Determine the sign of f' in each interval.
4. If it is positive, the function is increasing; if it is negative, the function is decreasing.
5. These guidelines are valid for infinite intervals.

Summary:

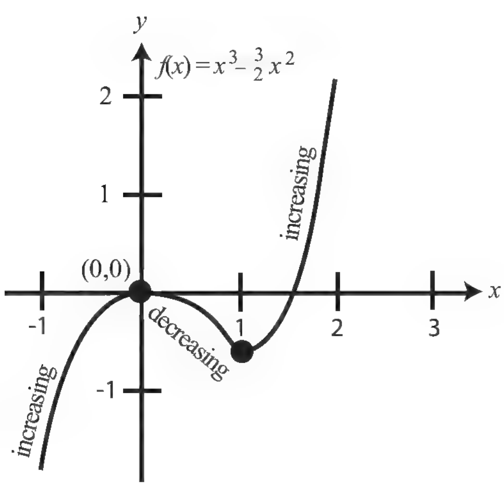
In general, the graph of a function is increasing (rising) if its derivative is positive and decreasing (falling) if its derivative is negative.

Example 1: Intervals on Which a Function Is Increasing or Decreasing

Find the open intervals on which the function $f(x)=x^3-\dfrac{3}{2}x^2$ is increasing or decreasing.

Solution:

Find the critical number by calculating the derivative $f'(x)=3x^2-3x=3x(x-1)$. Hence the critical numbers are 0 and 1. These 2 numbers determine 3 test intervals: $(-\infty,0)$, $(0,1)$, and $(1,\infty)$. For the interval $(-\infty,0)$, the test value $x=-1$ gives $f'(-1)=3(-1)(-1-1)>0$, which shows that the derivative is positive on that interval. Similarly, the derivative is negative on $(0,1)$ and positive on $(1,\infty)$. We conclude that f is increasing on the intervals $(-\infty,0)$ and $(0,1)$ and decreasing on $(0,1)$.



Notice in this example that there is a relative maximum at (0, 0) and a relative minimum at $\left(1,-\dfrac{1}{2}\right)$. This is the essence of the first derivative test.

Example 2: The First Derivative Test

Use the first derivative test to find the relative extrema of $f(x)=\dfrac{1}{2}x-\sin x$ on the interval $(0,2\pi)$.

Solution:

We have $f'(x)=\frac{1}{2}-\cos x=0$. The critical numbers are the solutions to the trigonometric equation $\cos x=\frac{1}{2}$ on the interval $(0,2\pi)$, which are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Analyzing the sign of f' on the 3 intervals, we see that f is decreasing on $\left(0,\frac{\pi}{3}\right)$ and increasing on $\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$. Therefore, the first derivative test tells us that there is a relative minimum at $x=\frac{\pi}{3}$. A similar argument shows that there is a relative maximum at $x=\frac{5\pi}{3}$.

Rolle's theorem and the mean value theorem are important theoretical tools in calculus. You should draw a picture to illustrate their geometric significance.

Example 3: The Mean Value Theorem

Verify the mean value theorem for the function $f(x)=5-\frac{4}{x}=5-4x^{-1}$, $1\le x\le 4$.

Solution:

The function is continuous on the closed interval and differentiable on the open interval. Furthermore, $\frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{4-1}{3}=1$, and $f'(x)=4x^{-2}=\frac{4}{x^2}$. Setting these equal to each other, $1=\frac{4}{x^2}$, which gives $x=\pm 2$. However, -2 is not in the interval, so the number guaranteed by the mean value theorem is $c=2$.

Study Tips:

- One of the most difficult steps in applying the first derivative test is finding the values for which the derivative is equal to 0 or undefined. This might require some complicated precalculus or a calculator. For instance, Example 2 above required us to solve a trigonometric equation.
- You should use a graphing utility to confirm your results. Remember that the Advanced Placement examination requires graphing utilities.

Pitfalls:

- Keep in mind that the sign of f' might not change as you cross a critical number.
- Make sure the values you obtain for Rolle's theorem and the mean value theorem are in the open interval. (See Example 3 above.)

Problems:

1. Identify the open intervals on which the function $f(x)=x^2-2x-8$ is increasing or decreasing.

2. Identify the open intervals on which the function $g(x)=x\sqrt{16-x^2}$ is increasing or decreasing.

3. **a.** Find the critical numbers of $f(x) = 2x^3 + 3x^2 - 12x$ (if any).

   b. Find the open interval(s) on which the function is increasing or decreasing.

   c. Apply the first derivative test to identify all relative extrema.

   d. Use a graphing utility to confirm your results.

4. **a.** Find the critical numbers of $f(x) = \dfrac{x^2}{x^2 - 9}$ (if any).

   b. Find the open interval(s) on which the function is increasing or decreasing.

   c. Apply the first derivative test to identify all relative extrema.

   d. Use a graphing utility to confirm your results.

5. The electric power P in watts in a direct-current circuit with 2 resistors, R_1 and R_2, connected in parallel is $P = \dfrac{vR_1R_2}{(R_1 + R_2)^2}$, where v is the voltage. If v and R_1 are held constant, what resistance R_2 produces maximum power?

6. Explain why Rolle's theorem does not apply to the function $f(x) = 1 - |x - 1|$ on the interval $[0, 2]$.

7. Determine whether the mean value theorem can be applied to $f(x) = \sin x$ on the closed interval $[0, \pi]$. If the mean value theorem can be applied, find all values of c in the interval $(0, \pi)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

8. True or false? The sum of 2 increasing functions is increasing.

9. True or false? There is a relative maximum or minimum at each critical number.

10. The function $s(t) = t^2 - 7t + 10$ describes the motion of a particle along a line.

    a. Find the velocity function of the particle at any time $t \geq 0$.

    b. Identify the time interval(s) in which the particle is moving in a positive direction.

    c. Identify the time interval(s) in which the particle is moving in a negative direction.

    d. Identify the time(s) at which the particle changes direction.

Lesson Fourteen
Concavity and Points of Inflection

Topics:

- Concavity.
- Points of inflection.
- The second derivative test.

Definitions and Theorems:

- Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on I and **concave downward** if f' is decreasing on I.
- The graph is concave upward if the graph is above its tangent lines and concave downward if the graph is below its tangent lines.
- A point $(c, f(c))$ is a **point of inflection** (or inflection point) if the concavity changes at that point.
- The **second derivative test**: Let $f'(c)=0$ (c is a critical number of f). If $f''(c)>0$, then f has a relative minimum at c. If $f''(c)<0$, then f has a relative maximum at c.

Test for Concavity:

- Let f have a second derivative on the interval I.
- If $f''(x)>0$ for all x in I, then the graph of f is concave upward on I.
- If $f''(x)<0$ for all x in I, then the graph of f is concave downward on I.

Summary:

The second derivative gives information about concavity, or how a graph bends. You determine concavity much like you found the intervals where a graph was increasing or decreasing, except this time you use the second derivative.

Example 1: Determining Concavity

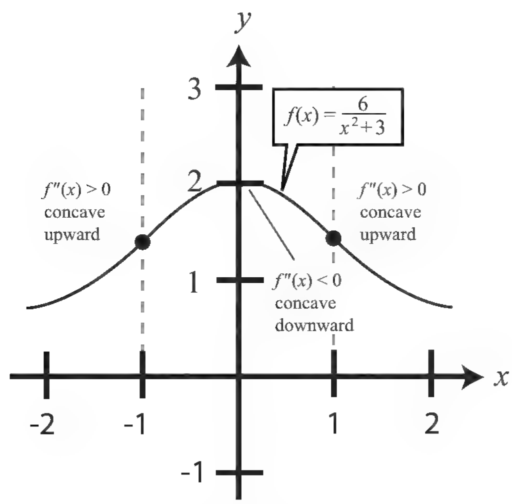
Find the open intervals on which the graph of the function $f(x)=\dfrac{6}{x^2+3}$ is concave upward or downward.

Solution:

Here are the first and second derivatives of the function. You should be able to verify these calculations.

$$f'(x)=\frac{-12x}{\left(x^2+3\right)^2} \qquad f''(x)=\frac{36(x+1)(x-1)}{\left(x^2+3\right)^3}$$

Because $f''(x)=0$ when $x=\pm 1$, we have 3 test intervals: $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. For the interval $(-\infty,-1)$, the test value $x=-2$ gives $f''(-2)>0$, which shows that the graph is concave upward on that interval. Similarly, the second derivative is negative on $(-1,1)$ and positive on $(1,\infty)$. We conclude that f is concave upward on the intervals $(-\infty,-1)$ and $(1,\infty)$ and concave downward on $(0,1)$.



In this example, note that the concavity changes at $x=\pm 1$. These are the points of inflection.

In the next example, we will see how the second derivative provides a convenient test for relative extrema. For instance, it is easy to see geometrically that if the graph is concave upward at a critical number, then you have a relative minimum.

Example 2: Using the Second Derivative Test

Use the second derivative test to find the relative extrema of $f(x)=-3x^5+5x^3$.

Solution:

We have $f'(x)=15x^2(1-x^2)$ and $f''(x)=30(x-2x^3)$. The critical numbers are 1, -1, and 0, and we evaluate the second derivative at these points. We see that $f''(1)<0$, so there is a relative maximum at 1, and $f''(-1)>0$, so there is a relative minimum at -1. Finally, $f''(0)=0$, so the second derivative test does not apply. By analyzing the first derivative, you see that 0 is not a relative extremum. You should verify these results with your graphing utility.

Study Tips:

- Two helpful examples are the basic functions $f(x)=x^2$ and $g(x)=x^3$. See the Summary Sheet in the appendix for examples. For example, note that the graph of f is concave upward for all x. The graph is above its tangent lines. Equivalently, the derivative $f'(x)=2x$ is increasing.
  A similar analysis can be applied to $g(x)=x^3$. In this case, you see that there is a point of inflection at the origin.
- Determining concavity is similar to finding the intervals where a graph is increasing or decreasing, except that you analyze the second derivative.
- If the second derivative is 0 at a critical point, then you cannot use the second derivative test.

Pitfalls:

- Even though $f''(c)=0$, keep in mind that the sign of f'' might not change as you cross this c value. For instance, $f''(0)=0$ for the function $f(x)=x^4$, but 0 is not a point of inflection.
- Keep in mind that points of inflection must be in the domain of the function. For example, the graph of $f(x)=\dfrac{1}{x}$ is concave downward on $(-\infty,0)$ and concave upward on $(0,\infty)$, but 0 is not a point of inflection.

Problems:

1. Determine the open intervals on which the function $f(x)=x^2-x-2$ is concave upward or concave downward.

2. Determine the open intervals on which the function $f(x)=3x^2-x^3$ is concave upward or concave downward.

3. Determine the open intervals on which the function $f(x)=2x-\tan x$, $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$, is concave upward or concave downward.

4. Find the points of inflection and discuss the concavity of the graph of $f(x)=-x^4+24x^2$.

5. Find the points of inflection and discuss the concavity of the graph of $f(x)=x(x-4)^3$.

6. Find the points of inflection and discuss the concavity of the graph of $f(x)=\sin x+\cos x$, $[0,2\pi]$.

7. Find all relative extrema of $f(x)=x^3-3x^2+3$. Use the second derivative test where applicable.

8. Find all relative extrema of $f(x)=x+\dfrac{4}{x}$. Use the second derivative test where applicable.

9. Discuss the concavity of the cube root function $f(x)=x^{\frac{1}{3}}$.

10. Discuss the concavity of the function $g(x)=x^{\frac{2}{3}}$.

Lesson Fifteen
Curve Sketching and Linear Approximations

Topics:

- Curve sketching.
- Linear approximations.
- Differentials.

Definitions and Theorems:

- Let $y = f(x)$ be a differentiable function. Then $dx = \Delta x$ is called the **differential** of x. The differential of y is $dy = f'(x)dx$.

Curve-Sketching Techniques:

Here are some concepts that are useful in analyzing the graph of a function.

1. Point plotting.
2. Symmetry.
3. Intercepts.
4. Domain and range.
5. Vertical and horizontal asymptotes.
6. Relative extrema.
7. Points of inflection.
8. Increasing or decreasing.
9. Concavity.

Summary:

You can use all of the skills we have developed so far to analyze and sketch the graph of a function. A graphing calculator (and point plotting) can help confirm that you have drawn the correct graph.

Example 1: Sketching the Graph of a Rational Function

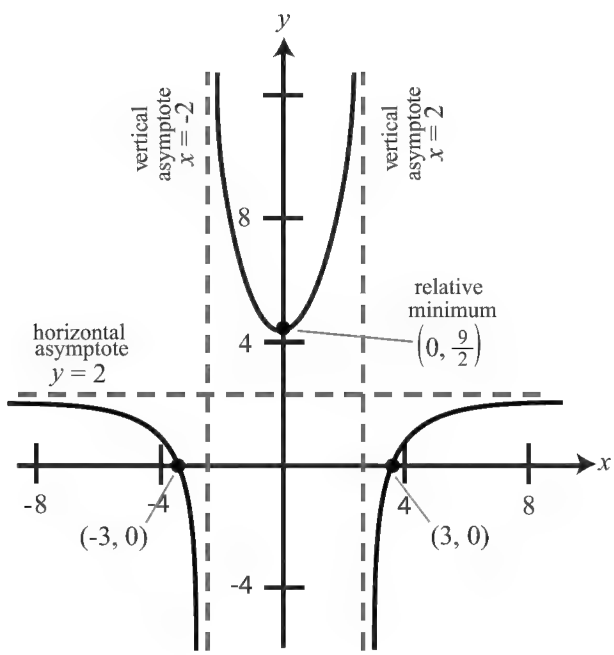
Analyze and sketch the graph of the function $f(x) = \dfrac{2(x^2-9)}{x^2-4}$.

Solution:

The intercepts are $(3,0)$, $(-3,0)$, and $\left(0, \dfrac{9}{2}\right)$. The domain is all $x \neq \pm 2$. The function is even, so the graph is symmetric about the y-axis. There are vertical asymptotes at $x = \pm 2$ and a horizontal asymptote at $y = 2$. Now we analyze the first and second derivatives of the function.

$$f'(x) = \frac{20x}{\left(x^2-4\right)^2} \qquad f''(x) = \frac{-20(3x^2+4)}{\left(x^2-4\right)^3}$$

The critical number is $x = 0$. Note that although $x = \pm 2$ are not critical numbers, they are used to form the test intervals. There are 4 test intervals for the first derivative, and we discover that the graph is decreasing on the intervals $(-\infty, -2)$ and $(-2, 0)$ and increasing on the intervals $(0, 2)$ and $(2, \infty)$. Similarly for the second derivative, we see that the graph is concave upward on $(-2, 2)$ and concave downward on $(-\infty, -2)$ and $(2, \infty)$. The point $\left(0, \frac{9}{2}\right)$ is a relative minimum.



Recall that the tangent line to a graph is a good approximation of the function near the point of tangency. This is the essence of linear approximation.

Example 2: Linear Approximation

Find the tangent line approximation to the function $f(x) = 1 + 2\sin x$ at the point $(0, 1)$.

Solution:

$f'(x) = 2\cos x,\ f'(0) = 2$, so the tangent line at $(0, 1)$ is

$$\begin{aligned} y - f(0) &= f'(0)(x - 0) \\ y - 1 &= 2(x - 0) \\ y &= 1 + 2x. \end{aligned}$$

You should verify that the tangent line is a good approximation to f near the point of tangency.

Study Tips:

- You might need to use precalculus to determine intercepts and critical numbers. For example, you can find the intercepts of $f(x)=2x^{\frac{5}{3}}-5x^{\frac{4}{3}}$ as follows:

$$\begin{aligned}&2x^{\frac{5}{3}}-5x^{\frac{4}{3}}=0\\&x^{\frac{4}{3}}(2x^{\frac{1}{3}}-5)=0\end{aligned}.$$

  Hence $x=0$ or $2x^{\frac{1}{3}}=5$. From the second equation, you obtain $x=\left(\frac{5}{2}\right)^3=\frac{125}{8}$. The intercepts are $(0,0)$ and $\left(\frac{125}{8},0\right)$.
- It is helpful to do some point plotting by hand, as well as with a graphing utility.

Pitfalls:

- A graphing utility can give misleading results. You will need calculus to determine where the important features of a graph are located. For example, graph the function f in Example 1 above on the standard viewing screen, along with the graph of $g(x)=\dfrac{2(x^2-9)(x-20)}{(x^2-4)(x-21)}$. Are all the important features of g apparent?
- Remember that critical numbers and points of inflection must be in the domain of the function.

Problems:

1. Analyze and sketch the graph of the function $f(x)=\dfrac{1}{x-2}-3$.
2. Analyze and sketch the graph of the function $f(x)=\dfrac{3x}{x^2-1}$.
3. Analyze and sketch the graph of the function $f(x)=3x^4+4x^3$.
4. Analyze and sketch the graph of the function $f(x)=x\sqrt{4-x^2}$.
5. Analyze and sketch the graph of the function $f(x)=2x-4\sin x,\ 0\le x\le 2\pi$.
6. Analyze and sketch the graph of the function $f(x)=2x-\tan x,\ -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$.
7. Find the tangent line approximation $T(x)$ to $f(x)=x^5$ at $(2,32)$. Compare the values of T and f for $x=1.9$ and $x=2.01$.
8. Find the tangent line approximation $T(x)$ to $f(x)=\sin x$ at $(2,\sin 2)$. Compare the values of T and f for $x=1.9$ and $x=2.01$.
9. Find the differential dy for $y=3x^2-4$.
10. Find the differential dy for $y=x\cos x$.

Lesson Sixteen
Applications—Optimization Problems, Part 1

Topic:

- Optimization problems.

Guidelines for Solving Applied Minimum and Maximum Problems:

1. Identify all given quantities and all quantities to be determined.
2. If possible, make a sketch.
3. Write a primary equation for the quantity to be maximized or minimized.
4. Reduce the primary equation to one with a single independent variable. This may involve the use of secondary equations relating the independent variables of the primary equation.
5. Determine the domain of the primary equation.
6. Use calculus to determine the desired maximum or minimum value.
7. Use calculus to verify the answer.

Summary:

One of the most common applications of calculus involves the determination of minimum and maximum values. We examine a typical example.

Example 1: Finding Maximum Volume

A manufacturer wants to design an open box with a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

Solution:

Draw the box and label the length of the square base x and the height h. The volume is given by the primary equation $V = x^2h.$ The surface area of the open box consists of the bottom and 4 sides: $x^2 + 4xh = 108.$ We can use this constraint to eliminate the variable h in the volume formula as follows:

$$4xh = 108 - x^2$$

$$h = \frac{108 - x^2}{4x}.$$

$$V = x^2h = x^2\left(\frac{108 - x^2}{4x}\right) = 27x - \frac{x^3}{4}.$$

The domain of this function is $0 \le x \le \sqrt{108}.$ We now use our calculus skills to find the maximum value of V.

$$\frac{dV}{dx} = 27 - \frac{3x^2}{4} = 0$$

$$3x^2 = 108$$

$$x = 6$$

The critical number is $x = 6.$ Since $V(0) = V(\sqrt{108}) = 0,$ $V(6) = 108$ is the maximum value on the interval. For $x = 6,$ you obtain $h = 3.$ The maximum volume is 108 cubic inches, and the dimensions are $6 \times 6 \times 3.$ Note that you could have used the first or second derivative tests to verify that 6 gave a maximum.

Study Tips:

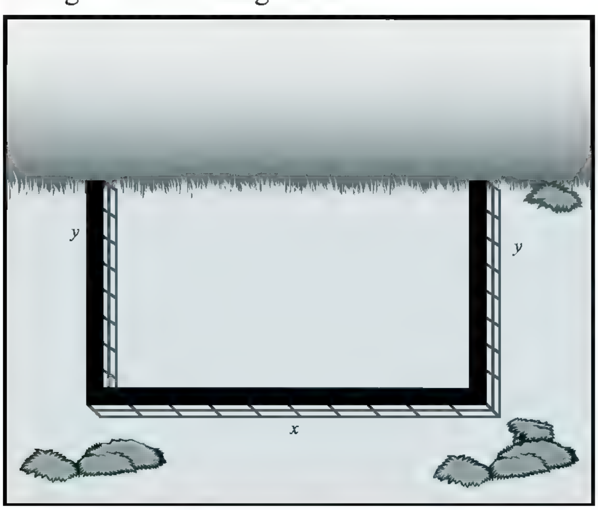
- When you solve an optimization problem, make sure that the critical number you obtain is indeed a maximum (or minimum). You can use the techniques for extrema on a closed interval (Lesson Twelve), the first derivative test (Lesson Thirteen), or the second derivative test (Lesson Fourteen).
- Optimization problems are difficult because they require formulas from precalculus, such as areas, volumes, and trigonometric relationships.

Pitfall:

- Remember that a maximum or minimum value can occur at an endpoint, so you need to evaluate the primary equation at the critical numbers and at the endpoints (if the interval is closed).

Problems:

1. Find 2 positive numbers such that their product is 185 and their sum is a minimum.
2. Find 2 positive numbers such that the second number is the reciprocal of the first number and their sum is a minimum.
3. Find the point on the graph of $f(x)=x^2$ that is closest to the point $\left(2,\frac{1}{2}\right)$.
4. Find the point on the graph of $f(x)=\sqrt{x}$ that is closest to the point $(4,0)$.
5. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 square meters to provide enough grass for the herd. What dimensions will require the least amount of fencing if no fencing is needed along the river?



6. The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and the square that produce a minimum total area.

Lesson Seventeen
Applications—Optimization Problems, Part 2

Topic:

- More optimization problems.

Guidelines for Solving Applied Minimum and Maximum Problems:

1. Identify all given quantities and all quantities to be determined.
2. If possible, make a sketch.
3. Write a primary equation for the quantity to be maximized or minimized.
4. Reduce the primary equation to one with a single independent variable. This may involve the use of secondary equations relating the independent variables of the primary equation.
5. Determine the domain of the primary equation.
6. Use calculus to determine the desired maximum or minimum value.
7. Use calculus to verify the answer.

Summary:

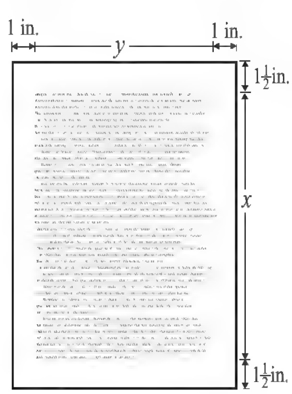
We look at more optimization examples in this second lesson on finding maximum and minimum values of functions.

Example 1: Finding a Minimum

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

Solution:

Use x and y to label the length and width of the printed portion of the page, as indicated in the figure.



Taking into account the margins, the area to be minimized is $A=(x+3)(y+2)$. The printed area inside the margins satisfies the equation $xy=24$. We can use this constraint to eliminate the variable y in the area formula, as follows:

$$xy=24$$
$$y=\frac{24}{x}.$$
$$A=(x+3)(y+2)=(x+3)\left(\frac{24}{x}+2\right)=30+2x+\frac{72}{x},\ x>0.$$

The domain of this function is $x>0$. We now use our calculus skills to find the minimum value of A.

$$\frac{dA}{dx}=2-\frac{72}{x^2}=0$$
$$2x^2=72$$
$$x^2=36$$
$$x=6$$

The critical number is $x=6$. Using the second derivative test, $A''(x)=\frac{144}{x^3}>0$, which shows that there is a minimum at $x=6$. The corresponding y value is $y=\frac{24}{6}=4$, and the dimensions of the page should be $(x+3)\times(y+2)$, or 9×6 inches.

Study Tip:

- When you solve an optimization problem, make sure that the critical number you obtain is indeed a maximum (or minimum). You can use the techniques for extrema on a closed interval (Lesson Twelve), the first derivative test (Lesson Thirteen), or the second derivative test (Lesson Fourteen).

Pitfall:

- Remember that a maximum or minimum value can occur at an endpoint, so you need to evaluate the primary equation at the critical numbers and at the endpoints (if the interval is closed). This was not an issue with the example above.

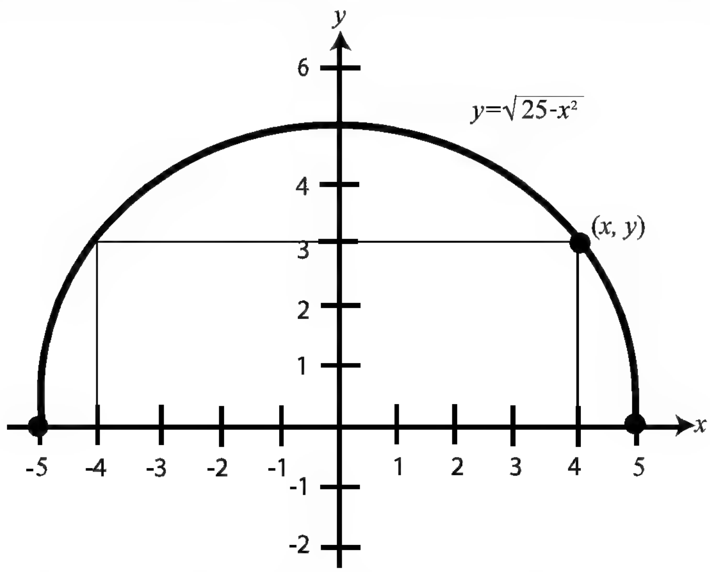
Problems:

1. A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

2. A rectangular page is to contain 36 square inches of print. The margins on each side are 1.5 inches. Find the dimensions of the page such that the least amount of paper is used.

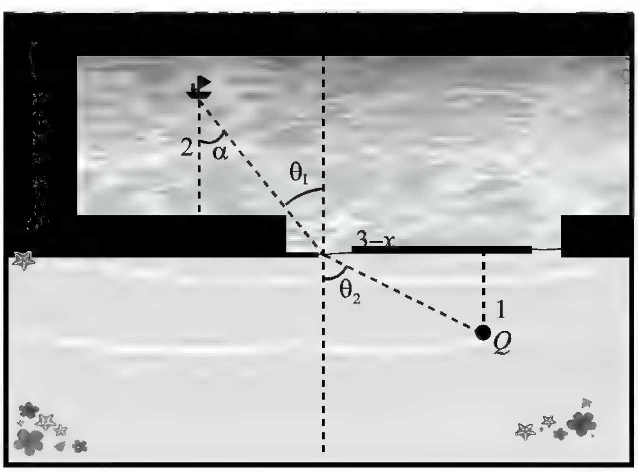
3. In an autocatalytic reaction, the product formed is a catalyst for the reaction. If Q_0 is the amount of the original substance and x is the amount of catalyst formed, the rate of the chemical reaction is $\frac{dQ}{dx}=kx(Q_0-x)$. For what value of x will the rate of chemical reaction be greatest?

4. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 5.



5. Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

6. A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland. He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row to reach point Q in the least time?



Lesson Eighteen
Antiderivatives and Basic Integration Rules

Topics:

- Antiderivatives, or integrals.
- Basic integration rules.
- Differential equations with initial conditions.
- Motion under the influence of gravity.

Definitions and Theorems:

- A function G is an **antiderivative** of f on an interval I if $G'(x) = f(x)$ for all x in I.
- Theorem: If G is an antiderivative of f, then so is $G + C$, where C is any constant.
- The position function of an object under the influence of gravity alone is $s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$. Here, g is the gravitational constant, v_0 is the initial velocity, and s_0 is the initial position.

Notation: $\int f(x)\,dx = G(x) + C$.

1. f is the integrand.
2. dx indicates that the variable of integration is x.
3. G is an antiderivative (or integral) of f.
4. C is the constant of integration
5. The expression is read as "the antiderivative of f with respect to x."

Basic Integration Rules Based on Derivative Formulas:

1. $\int 0\,dx = C.$
2. $\int k\,dx = kx + C.$
3. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$
4. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1.$
5. $\int \cos x\,dx = \sin x + C.$
6. $\int \sin x\,dx = -\cos x + C.$
7. $\int \sec^2 x\,dx = \tan x + C.$

Summary:

Antidifferentiation (or integration) is the inverse of differentiation. The antiderivative of $f(x) = 3x^2$ is $G(x) = x^3 + C$ because the derivative of G is f.

Example 1: Finding Antiderivatives

$$\int \cos x \, dx = \sin x + C$$
$$\int \left(x^5 + \sin x + 3\right) dx = \frac{x^6}{6} - \cos x + 3x + C$$

You can check that these are correct by differentiating the answers.

Example 2: Solving a Differential Equation

Solve the differential equation $G'(x) = \frac{1}{x^2}$, $x > 0$, that satisfies the initial condition $G(1) = 0$.

Solution:

$G(x) = \int \frac{1}{x^2} dx = \int x^{-2} \, dx = \frac{x^{-1}}{-1} + C = \frac{-1}{x} + C$. We can determine the constant of integration using the initial condition.

$$G(1) = \frac{-1}{1} + C = 0$$

Thus, $C = 1$, and the particular solution to the differential equation is $G(x) = \frac{-1}{x} + 1$, $x > 0$. Notice that the general solution of the differential equation has a constant of integration and represents a family of curves in the plane. The particular solution is one of these curves.

Study Tips:

- The power rule $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ is not valid for $n = -1$. We do not yet know how to find the antiderivative $\int \frac{1}{x} dx$. We will return to this question when we study logarithms.
- Finding antiderivatives is more difficult than calculating derivatives, but remember that you can always check your answer to an integral question by differentiating the result.

Pitfall:

- Some functions do not have antiderivatives among the functions that we use in elementary calculus. For example, you cannot solve $\int \cos(x^2) dx$. That is, there is no function G (among the functions in elementary calculus) whose derivative is $\cos(x^2)$.

Problems:

1. Find the integral: $\int (x + 7) dx$.
2. Find the integral: $\int (2x - 3x^2) dx$.
3. Find the integral: $\int \frac{x+6}{\sqrt{x}} dx$.

4. Find the integral: $\int \frac{x^2+2x-3}{x^4}\,dx$.
5. Find the integral: $\int (5\cos x + 4\sin x)\,dx$.
6. Find the integral: $\int \left(\theta^2 + \sec^2\theta\right) d\theta$.
7. Solve the differential equation $f'(x) = 6x,\ f(0) = 8$.
8. Solve the differential equation $h'(t) = 8t^3 + 5,\ h(1) = -4$.
9. A ball is thrown vertically from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?
10. With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?

Lesson Nineteen
The Area Problem and the Definite Integral

Topics:

- The area problem.
- Sigma notation.
- Summation formulas.
- The definite integral.
- Properties of the definite integral.

Definitions and Theorems:

- **Sigma notation for sums**: $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.
- Definition of the **area of a region in the plane**: Let f be continuous and nonnegative on the interval $[a,b]$. Partition the interval into n equal subintervals of length $\Delta x = \dfrac{b-a}{n}$, $x_0 = a,\ x_1,\ x_2,\ \ldots,\ x_{n-1}$, $x_n = b$. The area of the region bounded by f, the x-axis, and the vertical lines $x = a$ and $x = b$ is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,\ x_{i-1} \le c_i \le x_i$, provided this limit exists.

  Let f be defined on the interval $[a,b]$. Partition the interval into n equal subintervals of length $\Delta x = \dfrac{b-a}{n}$, $x_0 = a,\ x_1,\ x_2,\ \ldots,\ x_{n-1},\ x_n = b$. Assume that the following limit exists: $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$, where $x_{i-1} \le c_i \le x_i$. Then this limit is the **definite integral** of f from a to b and is denoted $\int_a^b f(x)\,dx$. The expression $\sum_{i=1}^{n} f(c_i)\Delta x$ is called a **Riemann sum**.

Properties:

- $\int_a^a f(x)\,dx = 0$; $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
- $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$; $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
- If f is nonnegative on $[a,b]$, then $0 \le \int_a^b f(x)\,dx$.
- If $f(x) \le g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Summation Formulas:

1. $\sum_{i=1}^{n} c = c + c + \cdots + c = cn$.
2. $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$.
3. $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Summary:

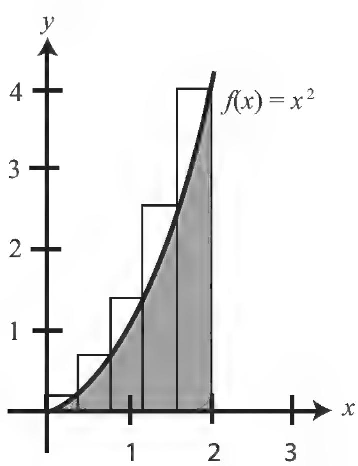
The area problem is much older than the tangent line problem. Even the early Greeks were able to find the area under simple curves defined on a closed interval. In the first example, we find the area under a parabola.

Example 1: The Area under a Parabola

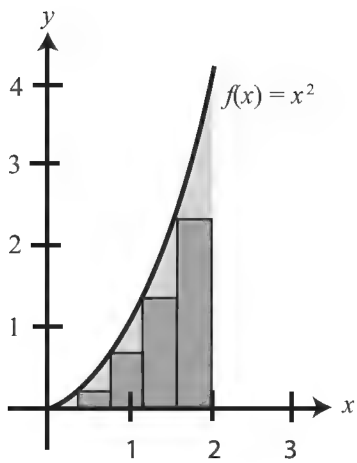
Calculate the area under the parabola $f(x) = x^2$ and above the x-axis, where $0 \le x \le 2$.

Solution:

Begin by partitioning the interval $[0,2]$ into n subintervals of length $\Delta x = \dfrac{2-0}{n}$. Use the right endpoints of each subinterval to determine the heights of the circumscribed rectangles, as indicated in the figure.



Circumscribed rectangles



Inscribed rectangles

The sum $S(n)$ of the areas of these n rectangles is called an **upper sum** and can be calculated using our summation formulas.

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\Delta x \\ &= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right) i^2 \\ &= \left(\frac{8}{n^3}\right) \sum_{i=1}^{n} i^2 \end{aligned}$$

We now use our summation formulas to evaluate this summation.

$$S(n) = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{4}{3n^2}\left(2n^2+3n+1\right) = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

As n tends to infinity, $S(n)$ approaches $\frac{8}{3}$. Similarly, using inscribed rectangles, you obtain the **lower sum** $s(n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}$, which also approaches $\frac{8}{3}$. Hence, we say that the area bounded by the parabola is $\frac{8}{3}$.

You can see that using the limit definition of area, or the corresponding definition of the definite integral, is difficult. We will see a more convenient method in the next lesson. But always keep in mind the summing rectangles model; this will be helpful when we use definite integrals in applications.

Of course, we can find some definite integrals easily if they correspond to geometric formulas, as follows.

Example 2: The Definite Integral

The region in the first quadrant bounded by $f(x)=4,\ 1\le x\le 3$, is a rectangle of area $4\times 2=8$. Using our notation for the definite integral, we have $\int_1^3 4\,dx=8$.

Study Tips:

- In the summation formulas, you can use any letter for the index of summation. You can find other summation formulas in a calculus textbook.
- For definite integrals, the variable in the integrand can be any letter. For example, the following are equivalent: $\int_1^3 x^2\,dx \;=\; \int_1^3 t^2\,dt$.
- In the area problem and the definite integral, letting n tend to infinity is equivalent to letting $\Delta x\to 0$.
- The integral sign $(\int)$ is an elongated letter S, which reminds us that we are summing rectangles of width Δx.

Pitfalls:

- Although you can move a constant outside the integral sign, you cannot move variables outside the integral sign.
- The integral of a sum is the sum of the integrals. However, the integral of a product is not the product of the integrals.

Problems:

1. Find the sum: $\sum_{i=1}^{6}(3i+2)$.
2. Find the sum: $\sum_{k=0}^{4}\frac{1}{k^2+1}$.
3. Use sigma notation to write the sum $\frac{1}{5(1)}+\frac{1}{5(2)}+\frac{1}{5(3)}+\cdots+\frac{1}{5(11)}$.
4. Use sigma notation to write the sum $\left[2\left(1+\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)+\cdots+\left[2\left(1+\frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$.
5. Use the formulas for summations to evaluate $\sum_{i=1}^{24}4i$.
6. Use the formulas for summations to evaluate $\sum_{i=1}^{20}(i-1)^2$.
7. Use $n=4$ rectangles and right endpoints to approximate the area of the region bounded by $f(x)=2x+5,\ 0\le x\le 2,$ in the first quadrant.
8. Use $n=4$ rectangles and right endpoints to approximate the area of the region bounded by $f(x)=\cos x,\ 0\le x\le\frac{\pi}{2},$ in the first quadrant.
9. Use the limit definition to find the area of the region between $f(x)=-4x+5,\ 0\le x\le 1,$ and the x-axis.

10. Use the limit definition to find the area of the region between $f(x) = x^2 + 1$, $0 \le x \le 3$, and the x-axis.

11. Use a geometric formula to determine the definite integral: $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$.

Lesson Twenty
The Fundamental Theorem of Calculus, Part 1

Topics:

- The fundamental theorem of calculus.
- The mean value theorem for integrals.
- Average value.
- A proof of the fundamental theorem of calculus.

Definitions and Theorems:

- The **fundamental theorem of calculus**: Let f be continuous on the closed interval $[a,b]$. Let G be an antiderivative of f. Then $\int_a^b f(x)\,dx = G(b)-G(a)$.
- Let f be continuous on the closed interval $[a,b]$. The **mean value theorem for integrals** says that there exists a number c in $[a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$. The value $f(c)$ is called the **average value** of f.

Summary:

The fundamental theorem of calculus permits us to evaluate definite integrals using antiderivatives.

Example 1: Evaluating a Definite Integral

Evaluate the definite integral $\int_1^2 \left(x^2-3\right)dx$.

Solution:

The function $G = \dfrac{x^3}{3} - 3x$ is an antiderivative of $f(x) = x^2 - 3$. We now evaluate G at the 2 endpoints and subtract. Notice the convenient notation:

$$\int_1^2 \left(x^2-3\right)dx = \left[\frac{x^3}{3}-3x\right]_1^2 = \left(\frac{8}{3}-6\right)-\left(\frac{1}{3}-3\right) = \frac{7}{3}-3 = -\frac{2}{3}.$$

Here are some more examples. Notice that you need to be able to find antiderivates to use the fundamental theorem of calculus.

Example 2: Evaluating Definite Integrals

$$\int_0^{\pi} \cos x\,dx = \left[\sin x\right]_0^{\pi} = \sin\pi - \sin 0 = 0$$

$$\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \left[\tan x\right]_0^{\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) - \tan 0 = 1$$

Example 3: The Mean Value Theorem for Integrals

Verify the mean value theorem for integrals for the function $f(x)=\sqrt{x}$ on the interval $[1,4]$.

Solution:

The theorem states that there exists a number c in the interval such that $\int_1^4 \sqrt{x}\,dx \;=\; f(c)(4-1)$.

We first evaluate the integral on the left: $\int_1^4 \sqrt{x}\,dx \;=\; \int_1^4 x^{\frac{1}{2}}\,dx \;=\; \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 \;=\; \frac{2}{3}(8-1) \;=\; \frac{14}{3}$.

Hence we have $\frac{14}{3}=\sqrt{c}(3)$, which gives $\sqrt{c}=\frac{14}{9}$ and $c=\left(\frac{14}{9}\right)^2 \approx 2.42$. The average value of f is $f(c)=\frac{14}{9}$.

Study Tips:

- The fundamental theorem of calculus depends on your ability to find antiderivatives.
- When finding an antiderivative for f, you do not need to include the constant of integration C. Any antiderivative will work.
- Remember that for definite integrals, the variable in the integrand (sometimes called a dummy variable) can be any letter. For example, the following are equivalent: $\int_a^b f(x)\,dx \;=\; \int_a^b f(t)\,dt \;=\; \int_a^b f(\theta)\,d\theta$.
- Like the mean value theorem for derivatives, the mean value theorem for integrals is an existence theorem. It does not say how to find the value c.
- If you are dealing with absolute value functions, you might have to split up the interval of integration. For example, $\int_0^4 |x-3|\,dx \;=\; \int_0^3 |x-3|\,dx \;+\; \int_3^4 |x-3|\,dx \;=\; \int_0^3 (3-x)\,dx \;+\; \int_3^4 (x-3)\,dx$.
- Keep in mind that a definite integral is a number, whereas an antiderivative is a function.

Pitfalls:

- Calculating definite integrals can involve messy arithmetic. Be very careful with the negative signs and fractions.
- An area problem will never have a negative answer, but definite integrals can be positive, negative, or 0.

Problems:

1. Evaluate the definite integral: $\int_0^2 6x\,dx$.

2. Evaluate the definite integral: $\int_0^1 (2t-1)^2\,dt$.

3. Evaluate the definite integral: $\int_1^4 \frac{u-2}{\sqrt{u}}\,du$.

4. Evaluate the definite integral: $\int_0^5 |2x-5|\,dx$.

5. Evaluate the definite integral of the trigonometric function: $\int_0^{\pi}(1+\sin x)\,dx$.

6. Evaluate the definite integral of the trigonometric function: $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,dx$.

7. Find the area of the region bounded by the graphs of the equations $y=5x^2+2,\ x=0,\ x=2,$ and $y=0$.

8. Find the area of the region bounded by the graphs of the equations $y=x^3+x,\ x=2,$ and $y=0$.

9. Find the value(s) of c guaranteed by the mean value theorem for integrals for the function $f(x)=x^3,\ 0\le x\le 3$.

10. Find the value(s) of c guaranteed by the mean value theorem for integrals for the function $f(x)=\cos x,\ -\dfrac{\pi}{3}\le x\le\dfrac{\pi}{3}$.

11. Find the average value of the function $f(x)=4x^3-3x^2$ on the interval $[-1,2]$.

Lesson Twenty-One
The Fundamental Theorem of Calculus, Part 2

Topics:

- The definite integral as a function.
- The second fundamental theorem of calculus.
- Net change.

Definitions and Theorems:

- A function of the form $G(x) = \int_a^x f(t)\,dt$ is called an **accumulation function**.
- Let f be continuous on an open interval I containing a. The **second fundamental theorem of calculus** says that for any x in the interval, $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.
- The fundamental theorem of calculus can be interpreted as measuring the **net change**, or **total change**: $\int_a^b G'(x)\,dx = G(b) - G(a)$.
- For a moving particle with position function $s(t)$ and velocity $s'(t) = v(t)$, the net change or displacement is $\int_a^b s'(t)\,dt = \int_a^b v(t)\,dt = s(b) - s(a)$. The **total distance traveled** is $\int_a^b |s'(t)|\,dt = \int_a^b |v(t)|\,dt$.

Summary:

You can consider a definite integral as an accumulation function by letting the upper limit of integration be a variable.

Example 1: The Definite Integral as a Function

Let $G(x) = \int_0^x \cos t\,dt$, and evaluate $G(0)$, $G\left(\frac{\pi}{6}\right)$, and $G\left(\frac{\pi}{2}\right)$.

Solution:

$$G(0) = \int_0^0 \cos t\,dt = 0$$

$$G\left(\frac{\pi}{6}\right) = \int_0^{\frac{\pi}{6}} \cos t\,dt = [\sin t]_0^{\frac{\pi}{6}} = \frac{1}{2}$$

$$G\left(\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \cos t\,dt = [\sin t]_0^{\frac{\pi}{2}} = 1$$

Notice that as x increases from 0 to $\frac{\pi}{2}$, the integral accumulates the area under the cosine curve.

Next, notice how the second fundamental theorem of calculus shows the inverse relationship between differentiation and integration.

Example 2: Using the Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int_1^x 4t^3\,dt\right] = 4x^3$$
$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}$$
$$\frac{d}{dx}\left[\int_x^3 \sin t^2\,dt\right] = \frac{d}{dx}\left[-\int_3^x \sin t^2 dt\right] = -\sin x^2$$

Example 3: Particle Motion

A particle is moving along a line so that its velocity is $v(t) = t^2 - 5t + 4$ feet per second. What is the displacement on the interval $0 \le t \le 5$? What is the total distance traveled on this interval?

Solution:

By factoring the velocity function $v(t) = (t-1)(t-4)$, you see that the particle is moving forward on the intervals $(0,1)$ and $(4,5)$ and moving backward on $(1,4)$. The displacement is given by

$$\int_0^5 v(t)\,dt \;=\; \int_0^5 \left(t^2 - 5t + 4\right)dt \;=\; \left[\frac{t^3}{3} - \frac{5t^2}{2} - 4t\right]_0^5 \;=\; \frac{125}{3} - \frac{125}{2} - 20 \;=\; -\frac{5}{6}.$$

To calculate the total distance, you have to break up the integral into 3 parts:

$$\int_0^5 |v(t)|\,dt \;=\; \int_0^1 v(t)\,dt \;+\; \int_1^4 -v(t)\,dt \;+\; \int_4^5 v(t)\,dt \;=\; \frac{11}{6}+\frac{9}{2}+\frac{11}{6} \;=\; \frac{49}{6}.$$

Study Tips:

- The fundamental theorem of calculus and the second fundamental theorem of calculus illustrate how differentiation and integration are related; loosely speaking, they are inverse operations of each other.
- To calculate the net change or displacement, you integrate the derivative. The answer could be positive, zero, or even negative. To calculate the total distance traveled, you integrate the absolute value of the derivative. The answer will always be nonnegative.

Pitfalls:

- For you to use the second fundamental theorem of calculus, the integral must be in the exact form given in the theorem. That is, the lower limit of integration is a constant, and the upper limit is the independent variable x.
- You can never have a definite integral with the same variable appearing in the integrand and in the limits of integration. For instance, it is incorrect to write $\int_3^x \sin x\,dx$.
- If you are using the fundamental theorem of calculus, make sure that the integrand is continuous on the interval under consideration. For example, you cannot apply the fundamental theorem of calculus to the integral $\int_{-1}^1 x^{-2}\,dx$ because the integrand is not continuous at $x = 0$.

Problems:

1. Let $F(x) = \int_0^x (4t-7)\,dt$, and evaluate F at $x=2$, $x=5$, and $x=8$.

2. Let $F(x) = \int_2^x \frac{-2}{t^3}\,dt$, and evaluate F at $x=2$, $x=5$, and $x=8$.

3. **a.** Integrate the function $F(x)=\int_0^x (t+2)\,dt$ to find F as a function of x.

b. Illustrate the second fundamental theorem of calculus by differentiating the result from part a.

4. **a.** Integrate the function $F(x)=\int_{\pi/4}^x \sec^2 t\,dt$ to find F as a function of x.

b. Illustrate the second fundamental theorem of calculus by differentiating the result from part a.

5. Use the second fundamental theorem of calculus to find $F'(x)$ if $F(x)=\int_0^x t\cos t\,dt$.

6. Use the second fundamental theorem of calculus to find $F'(x)$ if $F(x)=\int_1^x \sqrt[4]{t}\,dt$.

7. Find $F'(x)$ if $F(x)=\int_x^2 t^3\,dt$.

8. Find $F'(x)$ if $F(x)=\int_0^{\sin x} \sqrt{t}\,dt$.

9. Find $F'(x)$ if $F(x)=\int_x^{x+2} (4t+1)\,dt$.

10. A particle is moving along a line with velocity function $v(t)=t^2-t-12$ feet per second. Find the displacement on the interval $1\le t\le 5$ and the total distance traveled on that interval.

Lesson Twenty-Two
Integration by Substitution

Topics:

- Using integration by substitution to find antiderivatives.
- Using change of variables to evaluate definite integrals.
- Evaluating integrals of even and odd functions.

Definitions and Theorems:

- **Integration by substitution:** Let F be an antiderivative of f. If $u = g(x)$, then $du = g'(x)dx$, so we have
$$\int f(g(x))\,g'(x)\,dx \;=\; F(g(x)) + C \text{ because}$$
$$\int f(u)\,du \;=\; F(u) + C.$$
- The function f is **even** if $f(-x) = f(x)$. In this case, you have $\int_{-a}^{a} f(x)\,dx \;=\; 2\int_{0}^{a} f(x)\,dx.$
- The function f is **odd** if $f(-x) = -f(x)$. In this case, you have $\int_{-a}^{a} f(x)\,dx \;=\; 0.$

Summary:

In this lesson, we look at a technique for finding antiderivatives called integration by substitution. It is based on the chain rule, as illustrated in the following example.

Example 1: Integration by Substitution

Find $\int \cos(x^2)(2x)\,dx.$

Solution:

Let $u = x^2$, and hence $\frac{du}{dx} = 2x$, or $du = 2x\,dx$. By replacing x with the new variable u, we have
$$\int \cos(x^2)(2x)\,dx \;=\; \int \cos u\,du \;=\; \sin u + C.$$
Returning to the original variable x, the answer is
$$\int \cos(x^2)(2x)\,dx \;=\; \sin(x^2) + C.$$
If you verify this answer by differentiation, you will see how the chain rule comes into play.

Example 2: Integration by Substitution

Find $\int \sqrt{2x-1}\,dx.$

Solution:

Let $u = 2x - 1,\ du = 2\,dx$. Then you have $\sqrt{2x-1} = \sqrt{u},\ dx = \frac{du}{2}$. Substituting into the integral, you obtain

$$\int \sqrt{2x-1}\,dx = \int \sqrt{u}\left(\frac{du}{2}\right) = \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2x-1)^{\frac{3}{2}} + C.$$

Example 3: Integration by Substitution for a Definite Integral

Find $\int_0^2 3x^2\sqrt{x^3+1}\,dx$.

Solution:

One way to solve this problem is to use substitution to find the antiderivative of the integrand and then evaluate it at the limits of integration. Another convenient way is to convert the entire integral to one involving the new variable u. To see this, let $u = x^3 + 1,\ du = 3x^2\,dx$. When $x = 0,\ u = 0^3 + 1 = 1$. And when $x = 2,\ u = 2^3 + 1 = 9$. So we have an equivalent integral, $\int_1^9 u^{\frac{1}{2}}\,du.$ You can verify that the final answer is $\frac{52}{3}$. Geometrically, the 2 definite integrals represent different regions in the plane, but their areas are the same.

Integrating even and odd functions over symmetric intervals is easy.

Example 4: Even and Odd Functions

Even function: $\int_{-3}^{3} x^4\,dx = 2\int_0^3 x^4\,dx = 2\left[\frac{x^5}{5}\right]_0^3 = \frac{2}{5}(3^5) = \frac{486}{5}.$

Odd function: $\int_{-3}^{3} x^3\,dx = \int_{-3}^{0} x^3\,dx + \int_0^3 x^3\,dx = \left[\frac{x^4}{4}\right]_{-3}^{0} + \left[\frac{x^4}{4}\right]_0^3 = \frac{-81}{4} + \frac{81}{4} = 0.$

Study Tips:

- Recall that each derivative formula gives rise to a corresponding integral formula.
- You can always check an integration calculation by differentiating your answer.
- Be aware that there can be many ways to write an answer. The form of your answer might differ from that of a friend or of the answer in the book. You might have to do some algebra to show that the answers are equivalent.
- Some functions do not have antiderivatives (among the functions we use in elementary calculus). For example, there is no function among the elementary functions with a derivative equal to $f(x) = \cos x^2$.

Pitfalls:

- Although definite integrals can be positive, negative, or even zero, remember that area is always nonnegative.
- Even though you can move constants outside the integral sign, you cannot do this with variables.
- Make sure your answer is expressed in the original variable of the problem. Do not leave the answer in the substitution variable u.

Problems:

1. Find the integral: $\int (1+6x)^4 (6)\, dx$.

2. Find the integral: $\int \sqrt{25-x^2}\,(-2x)\, dx$.

3. Find the integral: $\int t\sqrt{t^2+2}\, dt$.

4. Find the integral: $\int \frac{x}{\left(1-x^2\right)^3}\, dx$.

5. Find the integral: $\int \sin 4x\, dx$.

6. Find the integral: $\int \tan^4 x \sec^2 x\, dx$.

7. Evaluate the definite integral: $\int_0^4 \frac{1}{\sqrt{2x+1}}\, dx$.

8. Evaluate the definite integral: $\int_0^{\pi/2} \cos\left(\frac{2x}{3}\right) dx$.

9. Evaluate the integral using the properties of even and odd functions: $\int_{-2}^{2} x^2\left(x^2+1\right) dx$.

10. Evaluate the integral using the properties of even and odd functions: $\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx$.

Lesson Twenty-Three
Numerical Integration

Topics:

- Numerical integration.
- The trapezoidal rule.
- Applications.

Definitions and Theorems:

- Let f be continuous on the interval $[a,b]$. Partition the interval into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$: $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. **The trapezoidal rule** for approximating the definite integral of f between a and b is given by
$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Summary:

In this lesson, we look at techniques to approximate definite integrals. We have already seen how to use inscribed and circumscribed rectangles, but using trapezoids is a better method.

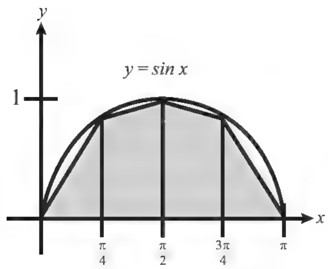
Example 1: The Trapezoidal Rule

Approximate the area under 1 arch of the sine curve using $n = 4$.

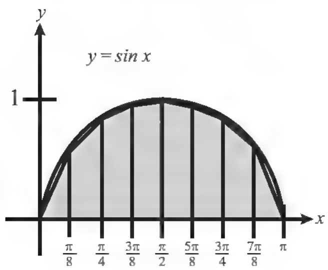
Solution:

In this case, $\Delta x = \dfrac{\pi - 0}{4} = \dfrac{\pi}{4}$, and we have
$$\begin{aligned}\int_0^\pi \sin x\,dx &= \frac{\pi - 0}{2(4)}\left[\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right]\\ &= \frac{\pi}{8}\left[0 + \sqrt{2} + 2 + \sqrt{2} + 0\right]\\ &= \frac{\pi(1+\sqrt{2})}{4}\\ &\approx 1.896.\end{aligned}$$

If we had used $n = 8$ subintervals, the answer would be approximately 1.974. If you use more subintervals, your approximation will get closer to the exact answer, which is 2.



Four subintervals



Eight subintervals

Study Tips:

- The fundamental theorem of calculus requires that you know the antiderivative of the integrand. On the other hand, numerical techniques can be used to approximate almost any definite integral, including integrals such as $\int_0^{\pi} \sin x^2 \, dx$.
- How accurate is the trapezoidal rule? This is a difficult question and best left for more advanced courses. However, it is clear that your answer will be better if you use more subintervals.
- There are many other approximation techniques: Riemann sums using left endpoints, Riemann sums using right endpoints, the midpoint rule, Simpson's rule, and so on. You can read about these methods in any calculus textbook. Moreover, most graphing utilities have built-in programs for approximating integrals.

Pitfall:

- Make sure the integral you are approximating is continuous on the closed interval. For example, you cannot use the trapezoidal rule for the integral $\int_{-1}^{1} \frac{1}{x^2} dx$ because the integrand is not continuous at 0.

Problems:

1. Use the trapezoidal rule and $n = 4$ to approximate $\int_0^2 x^3 dx$.

2. Use the trapezoidal rule and $n = 4$ to approximate $\int_0^2 x\sqrt{x^2+1}\, dx$.

3. Use the trapezoidal rule and $n = 8$ to approximate $\int_0^8 \sqrt[3]{x}\, dx$.

4. Approximate the integral $\int_0^2 \sqrt{1+x^3} dx$ with the trapezoidal rule and $n = 4$. Compare your answer with the approximation using a graphing utility.

5. Approximate the integral $\int_{3}^{3.1} \cos x^2 \, dx$ with the trapezoidal rule and $n = 4$. Compare your answer with the approximation using a graphing utility.

6. The elliptic integral $8\sqrt{3}\int_{0}^{\pi/2} \sqrt{1-\frac{2}{3}\sin^2\theta}\, d\theta$ gives the circumference of an ellipse. Approximate this integral with a numerical technique.

7. Later in this course, you will study inverse trigonometric functions. You will learn that π can be expressed as a definite integral: $\pi = \int_{0}^{1} \frac{4}{1+x^2} dx$. Use a numerical technique to approximate this integral, and compare your answer to π.

8. Suppose f is concave downward on the interval $[0,2]$. Using the trapezoidal rule, would your answer be too large or too small compared to the exact answer?

Lesson Twenty-Four
Natural Logarithmic Function—Differentiation

Topics:

- The natural logarithmic function.
- Properties of the logarithmic function.
- The derivative of the logarithmic function.
- Logarithmic differentiation.

Definitions and Theorems:

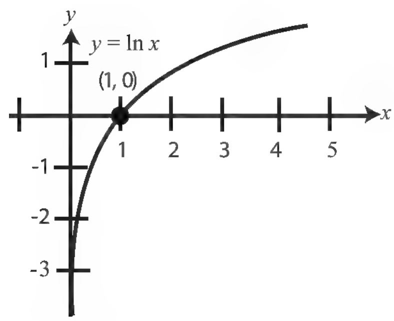
- The **natural logarithmic function** is defined by a definite integral: $\ln x = \int_1^x \frac{1}{t}\,dt,\ x > 0$.
- The derivative of the natural logarithmic function: $\frac{d}{dx}[\ln x] = \frac{1}{x}$.
- **Log rule for integration**: $\int \frac{1}{x}\,dx = \ln|x| + C$.
- The irrational number $e \approx 2.7182818$ is defined to be the unique number satisfying $\int_1^e \frac{1}{t}\,dt = 1$; that is, $\ln e = 1$.

Properties:

- The domain of the natural logarithmic function is $x > 0$, and the range is all real numbers.
- The graph of $y = \ln x$ is increasing and concave downward.
- The logarithmic function is continuous and one-to-one.
- For $x > 1$, $\ln x > 0$.
- For $0 < x < 1$, $\ln x < 0$.
- The natural logarithmic function satisfies the usual properties of logarithms:
  1. $\ln 1 = 0$.
  2. $\ln(xy) = \ln x + \ln y$.
  3. $\ln \frac{x}{y} = \ln x - \ln y$.
  4. $\ln x^a = a \ln x$.

Summary:

In this lesson, we define the natural logarithmic function as a certain integral. By the second fundamental theorem of calculus, we immediately see that we have an antiderivative for $\frac{1}{x} = x^{-1}$: $\int \frac{1}{x}\,dx = \ln x + C$. You should memorize the shape of the graph of $y = \ln x$ below.



The logarithmic function obeys the familiar rules of logarithms, as illustrated in the next example.

Example 1: Using Logarithmic Properties

$$\ln 8 - \ln 2 = \ln \frac{8}{2} = \ln 4$$
$$\ln\left(\sqrt{3x+2}\right) = \ln\left(3x+2\right)^{\frac{1}{2}} = \frac{1}{2}\ln(3x+2)$$

Example 2: Derivatives of Logarithmic Functions

$$\frac{d}{dx}\left[\ln 2x\right] = \frac{1}{2x}(2) = \frac{1}{x}$$
$$\left[x\ln x\right]' = x\left(\frac{1}{x}\right) + \ln x(1) = 1 + \ln x$$
$$\left[\ln\left|\sin x\right|\right]' = \frac{1}{\sin x}(\cos x) = \cot x$$

Logarithms can help to simplify derivative calculations, as follows.

Example 3: Logarithmic Differentiation

Use logarithmic differentiation to calculate the derivative of $y = x\sqrt{x^2+1}$ at $x = 1$.

Solution:

Take the logarithm of both sides of the equation.

$$\ln y = \ln\left[x\left(x^2+1\right)^{\frac{1}{2}}\right] = \ln x + \frac{1}{2}\ln\left(x^2+1\right)$$

Now differentiate both sides of the equation using implicit differentiation.

$$\frac{y'}{y} = \frac{1}{x} + \frac{1}{2}\left(\frac{1}{x^2+1}\right)(2x)$$
$$y' = y\left[\frac{1}{x} + \frac{x}{x^2+1}\right]$$

When $x = 1,\ y = \sqrt{2},$ and you have $y' = \sqrt{2}\left[1 + \frac{1}{2}\right] = \frac{3\sqrt{2}}{2}$.

Study Tips:

- The power rule $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$, was not valid for the case $\int \frac{1}{x}\,dx$. The natural logarithmic function fills in this gap.
- In precalculus, you studied logarithms to base 10. In calculus, we use the natural logarithm to base e. Graphing utilities have keys for both logarithms.

Pitfalls:

- A common error is to assume that the logarithm of a sum is the sum of the logarithms.
- Another common confusion arises when evaluating the logarithm of 1. Keep in mind that $\ln 1 = 0$, whereas $\ln 0$ is not defined.
- Remember that the domain of the logarithm function is all positive real numbers. Note the following error: $\ln 25 = \ln\left[(-5)(-5)\right] \neq \ln(-5) + \ln(-5)$.

Problems:

1. Use the properties of logarithms to expand the expression $\ln\frac{x}{4}$.

2. Use the properties of logarithms to expand the expression $\ln\left(x\sqrt{x^2+5}\right)$.

3. Write the expression $3\ln x + 2\ln y - 4\ln z$ as a logarithm of a single quantity.

4. Write the expression $2\ln 3 - \frac{1}{2}\ln(x^2+1)$ as a logarithm of a single quantity.

5. Find the derivative of the function $f(x) = \ln(3x)$.

6. Find the derivative of the function $f(x) = (\ln x)^4$.

7. Find the derivative of the function $f(x) = x^2 \ln x$.

8. Find the derivative of the function $f(x) = \ln(\ln x^2)$.

9. Find the derivative of the function $f(x) = \ln\left|\sec x + \tan x\right|$. Simplify your answer.

10. Find an equation of the tangent line to the graph of $y = \ln x^3$ at the point $(1,0)$.

11. Use implicit differentiation to find y' if $\ln xy + 5x = 30$.

12. Use logarithmic differentiation to find y' if $y = \sqrt{\dfrac{x^2-1}{x^2+1}},\ x > 1$.

13. Use your knowledge of graphing to graph the function $y = x\ln x$.

Lesson Twenty-Five
Natural Logarithmic Function—Integration

Topics:

- The natural logarithmic function and integration.
- The integrals of the trigonometric functions.
- Inverse functions.

Definitions and Theorems:

- **Log rule for integration**: $\int \frac{1}{x}\,dx = \ln|x| + C$.
- The integrals of the 6 trigonometric functions:

  1. $\int \sin x\,dx \;=\; -\cos x + C$.

  2. $\int \cos x\,dx \;=\; \sin x + C$.

  3. $\int \tan x\,dx \;=\; -\ln|\cos x| + C \;=\; \ln|\sec x| + C$.

  4. $\int \cot x\,dx \;=\; \ln|\sin x| + C$.

  5. $\int \sec x\,dx \;=\; \ln|\sec x + \tan x| + C$.

  6. $\int \csc x\,dx \;=\; -\ln|\csc x + \cot x| + C$.
- A function g is the **inverse function** of the function f if $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for all x in the domain of f. The inverse of f is denoted f^{-1}.

Properties:

- A function has an inverse if and only if the function is one-to-one.
- The slopes of inverse functions are reciprocals of each other.

Summary:

In this lesson, we explore the consequences of the integral formula $\int \frac{1}{x}\,dx = \ln|x| + C$.

Example 1: The Log Rule for Integration

$$\int \frac{2}{x}\,dx \;=\; 2\int \frac{1}{x}\,dx \;=\; 2\ln|x| + C \;=\; \ln x^2 + C$$

$$\int \frac{\sec^2 x}{\tan x}\,dx \;=\; \ln|\tan x| + C$$

Example 2: A Differential Equation

Solve the differential equation $\frac{dy}{dx} = \frac{1}{x\ln x}$.

Solution:

$$y = \int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \frac{1}{x} dx = \ln\left|\ln x\right| + C$$

You should verify this antiderivative by differentiating the answer. You will obtain the original integrand.

We are now able to calculate the integrals of all 6 trigonometric functions.

Example 3: The Integral of the Tangent Function

$$\int \tan x \, dx \;=\; \int \frac{\sin x}{\cos x} dx \;=\; -\int \frac{-\sin x}{\cos x} dx$$

Now let $u = \cos x,\ du = -\sin x \, dx$: $\int \tan x \, dx \;=\; -\ln\left|\cos x\right| + C \;=\; \ln\left|\sec x\right| + C$.

Study Tips:

- The integral of the secant function is complicated, but you can verify it by differentiating the answer. The integrals of the cotangent and cosecant functions are not very important.
- You are encouraged to check your results to integration problems by differentiating the answer.
- Long division of polynomials is often helpful in integrals involving quotients of polynomials. For instance, $\int \frac{x^2 + x + 1}{x^2 + 1} dx \;=\; \int \left(1 + \frac{x}{x^2 + 1}\right) dx$, and this latter integral is easier to solve.
- If f and g are inverses of each other, then their graphs are symmetric about the line $y = x$.

Pitfalls:

- The notation f^{-1} denotes the inverse of the function f, not the reciprocal $\frac{1}{f}$.
- The sine function is not one-to-one. To define the inverse sine function, you must restrict the domain of the sine function to the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Problems:

1. Find the indefinite integral: $\int \frac{1}{2x+5} dx$.

2. Find the indefinite integral: $\int \frac{x^2 - 4}{x} dx$.

3. Find the indefinite integral: $\int \frac{x^2 - 3x + 2}{x+1} dx$.

4. Find the indefinite integral: $\int \tan 5\theta \, d\theta$.

5. Find the indefinite integral: $\int \frac{\sec x \tan x}{\sec x - 1} dx$.

6. Solve the differential equation $\frac{dy}{dx} = \frac{3}{2-x}$.

7. Evaluate the definite integral: $\int_1^e \frac{(1+\ln x)^2}{x} dx$.

8. Evaluate the definite integral: $\int_0^1 \frac{x-1}{x+1} dx$.

9. Find the area of the region bounded by the equations $y = \frac{x^2+4}{x}$, $x=1$, $x=4$, and $y=0$.

10. Find the average value of the function $f(x) = \frac{8}{x^2}$ over the interval $[2,4]$.

11. Show that $f(x) = \sqrt{x-4}$ and $g(x) = x^2 + 4$, $x \geq 0$, are inverse functions.

12. Does the function $f(x) = \tan x$ have an inverse function? Why or why not?

Lesson Twenty-Six
Exponential Function

Topics:

- The exponential function.
- Properties of the exponential function.
- The derivative and integral of the exponential function.

Definitions and Theorems:

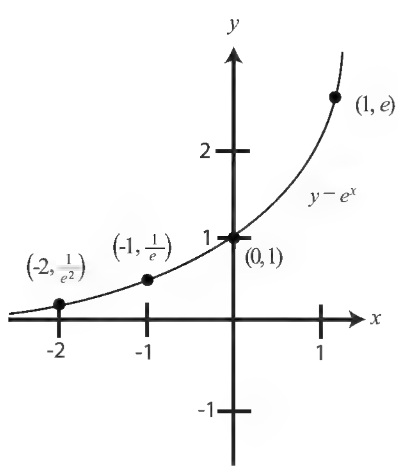
- The inverse of the natural logarithmic function $y = \ln x$ is the **exponential function** $y = e^x$.
- The exponential function is equal to its derivative: $\frac{d}{dx}\left[e^x\right] = e^x$.

Properties:

- $\ln(e^x) = x;\ e^{\ln x} = x$.
- $y = e^x$ if and only if $x = \ln y$.
- $e^a e^b = e^{a+b};\ \frac{e^a}{e^b} = e^{a-b}$.
- $\lim_{x \to \infty} e^x = \infty;\ \lim_{x \to -\infty} e^x = 0$.

Summary:

In this lesson, we define the exponential function $f(x) = e^x$ as the inverse of the natural logarithmic function. The domain of f is the set of all real numbers, and the range is the set of positive real numbers. The graph is continuous, increasing, one-to-one, and concave upward. You should memorize the shape of the graph of the exponential function; it is the mirror image of the graph of the natural logarithmic function across the line $y = x$.



Using the inverse relationship between these functions allows you to solve equations involving logarithms and exponentials.

Example 1: Solving an Equation

Solve for x in the equation $\ln(2x-3)=5$.

Solution:

One way to solve the equation is to take both sides to the power e.

$$\begin{aligned} \ln(2x-3) &= 5 \\ e^{\ln(2x-3)} &= e^5 \\ 2x-3 &= e^5 \\ 2x &= 3+e^5 \\ x &= \frac{1}{2}\left(3+e^5\right) \approx 75.707 \end{aligned}$$

Because the derivative of the exponential function is itself, you can use the chain rule to calculate other derivatives.

Example 2: Derivatives of Exponential Functions

$$\frac{d}{dx}\left[e^{2x-1}\right]=e^{2x-1}(2)=2e^{2x-1}$$

$$\frac{d}{dx}\left[e^x \ln x\right]=e^x\frac{1}{x}+\ln x(e^x)=e^x\left(\frac{1}{x}+\ln x\right)$$

The derivative formula $\left[e^x\right]'=e^x$ tells us that the antiderivative of e^x is simply e^x+C.

Example 3: Integrals of Exponential Functions

$$\int e^{3x+1}\,dx = \frac{1}{3}\int e^{3x+1}(3)\,dx = \frac{1}{3}e^{3x+1}+C$$

$$\int xe^{-x^2}\,dx = \frac{-1}{2}\int e^{-x^2}(-2x)\,dx = -\frac{1}{2}e^{-x^2}+C$$

Study Tips:

- Because $e\approx 2.718$, the graph of the exponential function $y=e^x$ lies between the graphs of $y=2^x$ and $y=3^x$.
- The inverse relationship of the exponential function and the natural logarithmic function is similar to that of logarithms to base 10: $y=10^x$ if and only if $x=\log_{10} y$.
- The derivative formula for the exponential function means that the slope of the graph at any point equals the y-coordinate at that point.

Pitfalls:

- Don't forget that $\ln e=1$ and $\ln 0$ is undefined.
- As shown in Example 3, you can find an antiderivative for $y=xe^{-x^2}$. However, there is no antiderivative for $y=e^{-x^2}$ among the elementary functions of calculus.

- Notice in Example 1 that the final decimal answer is an approximation. You should indicate this with the symbol $\approx$.

Problems:

1. Solve for x if $e^x = 12$.

2. Solve for x if $\ln x = 2$.

3. Solve for x if $\ln\sqrt{x+2} = 1$.

4. Find the derivative of the function $f(x) = e^{\sqrt{x}}$.

5. Find the derivative of the function $f(x) = \dfrac{e^x + 1}{e^x - 1}$.

6. Find the derivative of the function $f(x) = e^x(\sin x + \cos x)$.

7. Find the equation of the tangent line to the graph of $y = xe^x - e^x$ at the point $(1,0)$.

8. Find the indefinite integral: $\int x^2 e^{x^3}\, dx$.

9. Find the indefinite integral: $\int \dfrac{e^{-x}}{1+e^{-x}}\, dx$.

10. Evaluate the definite integral: $\int_3^4 e^{3-x}\, dx$.

11. Evaluate the definite integral: $\int_1^3 \dfrac{e^{3/x}}{x^2}\, dx$.

12. Use implicit differentiation to find $\dfrac{dy}{dx}$ if $xe^y - 10x + 3y = 0$.

13. Find the second derivative of the function $f(x) = (3+2x)e^{-3x}$.

Lesson Twenty-Seven
Bases other than *e*

Topics:

- Logarithmic and exponential functions to different bases.
- Properties of logarithmic functions and exponential functions.
- Derivatives and integrals of logarithmic and exponential functions.
- Another look at the number *e*.
- Compound interest.

Definitions and Theorems:

- The **exponential function to base *a***, $a > 0$, is defined by $a^x = e^{(\ln a)x}$.
- The **logarithmic function to base *a***, $a > 0$, $a \neq 1$, is defined by $\log_a x = \dfrac{1}{\ln a} \ln x$.
- Derivatives for bases other than *e*:
  $$\frac{d}{dx}\left[a^x\right] = (\ln a)a^x \,;\ \int a^x \, dx = \frac{1}{\ln a} a^x.$$
  $$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{\ln a}\frac{1}{x} = \frac{1}{(\ln a)x}.$$
- Integral for bases other than *e*:
  $$\int a^x \, dx = \frac{1}{\ln a} a^x .$$
- A limit involving *e*:
  $$e = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x .$$
- **Compound interest formulas**: Let P be the amount of a deposit at an annual interest rate of r (as a decimal) compounded n times per year. The amount after t years is $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$. If the interest is compounded continuously, the amount is $A = Pe^{rt}$.

Properties:

- $\log_a (xy) = \log_a x + \log_a y$.
- $\log_a \dfrac{x}{y} = \log_a x - \log_a y$.
- $\log_a 1 = 0;\ \log_a a = 1$.
- $\log_a x^n = n \log_a x$.

Summary:

In this lesson, we extend the definitions of the logarithmic and exponential functions to arbitrary bases $a > 0$. These functions satisfy the usual algebraic properties of precalculus logarithms and exponents. Using the inverse relationship between these functions allows you to solve equations involving logarithms and exponentials.

Example 1: Solving an Equation

Solve for x in the equation $3^x = \frac{1}{81}$.

Solution:

One way to solve the equation is to take logarithms to base 3.

$$\begin{aligned} 3^x &= \frac{1}{81} \\ \log_3(3^x) &= \log_3 \frac{1}{81} \\ x &= \log_3\left(3^{-4}\right) = -4 \end{aligned}$$

The derivative formulas for logarithmic and exponential functions are more complicated when the base is not e.

Example 2: Derivatives of Logarithmic and Exponential Functions

$$\frac{d}{dx}\left[2^x\right] = (\ln 2)2^x$$

$$\frac{d}{dx}\left[\log_{10} x\right] = \frac{1}{(\ln 10)x}$$

$$\frac{d}{dx}\left[\log_{10} \cos x\right] = \frac{1}{(\ln 10)\cos x}(-\sin x) = -\frac{\tan x}{\ln 10}$$

The derivative formula $\left[a^x\right]' = (\ln a)a^x$ tells us that the antiderivative of a^x is $\left(\frac{1}{\ln a}\right)a^x + C$. Notice again that the formula is simpler using base e.

Example 3: Area

Find the area under the curve $y = 2^x$ between $x = 0$ and $x = 2$.

Solution:

$$\text{Area} = \int_0^2 2^x\,dx = \left[\frac{1}{\ln 2}2^x\right]_0^2 = \frac{1}{\ln 2}\left(2^2 - 2^0\right) = \frac{3}{\ln 2}.$$

The formula for continuous compound interest is derived from the limit definition of the number e.

Example 4: Compound Interest

Suppose \$2500 is deposited at 5% interest for 5 years.

a. How much will there be in the account if the interest is compounded quarterly?

b. How much will there be if the interest is compounded continuously?

Solution:

a. For $n=4,\ P=2500,\ t=5,\ r=0.05$, you have $A=2500\left(1+\dfrac{0.05}{4}\right)^{4(5)}\approx \$3205.09$.

b. For continuous compounding, $A=2500e^{0.05(5)}\approx \$3210.06$.

Study Tips:

- The most convenient base is the number e. Notice how the definitions of the exponential and logarithmic functions to base a greatly simplify if $a=e$.
- The functions $f(x)=\log_a x$ and $g(x)=a^x$ are inverses of each other. Their graphs are symmetric about the line $y=x$. Furthermore, $y=a^x$ if and only if $x=\log_a y$.
- The definition of the logarithm to base a should remind you of the change of base formula for logarithms in precalculus. You can use this formula to evaluate logarithms to any base on a calculator.
- In many textbooks, the number e is defined by the following limit: $e=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$.

Pitfalls:

- A common error is to assume that the logarithm of a sum is the sum of the logarithms: $\log_a(x+y)\neq \log_a x+\log_a y$.
- Although $\int \frac{1}{x}dx = \ln x+C$, we do not yet have a formula for the integral of $f(x)=\ln x$ nor $f(x)=\log_a x$. This will be developed in Lesson Thirty-Four.

Problems:

1. Solve for x if $\log_3 x=-1$.

2. Solve for x if $x^2-x=\log_5 25$.

3. Solve for x if $3^{2x}=75$.

4. Find the derivative of the function $f(x)=5^{-4x}$.

5. Find the derivative of the function $f(x)=x9^x$.

6. Find the derivative of the function $f(x)=\log_5\sqrt{x^2-1}$.

7. Find the equation of the tangent line to the graph of $y=\log_{10}2x$ at the point $(5,1)$.

8. Find the indefinite integral: $\int 5^{-x}\,dx$.

9. Find the indefinite integral: $\int\left(x^3+3^{-x}\right)dx$.

10. Evaluate the definite integral: $\int_1^e\left(6^x-2^x\right)dx$.

11. You deposit $\$1000$ in an account that pays 5% interest for 30 years. How much will be in the account if the interest is compounded

a. monthly?

b. continuously?

12. Use logarithmic differentiation to find $\frac{dy}{dx}$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $y = x^{\sin x}$.

Lesson Twenty-Eight
Inverse Trigonometric Functions

Topics:

- The inverse sine function.
- The inverse trigonometric functions.
- Derivatives of inverse trigonometric functions.
- Integral formulas involving inverse trigonometric functions.

Definitions:

- The **inverse trigonometric functions** are defined as follows:

  $y = \arcsin x = \sin^{-1} x \Leftrightarrow \sin y = x,$ for $-1 \le x \le 1$ and $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

  $y = \arccos x = \cos^{-1} x \Leftrightarrow \cos y = x,$ for $-1 \le x \le 1$ and $0 \le y \le \pi$.

  $y = \arctan x = \tan^{-1} x \Leftrightarrow \tan y = x,$ for $-\infty < x < \infty$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

  $y = \operatorname{arcsec} x = \sec^{-1} x \Leftrightarrow \sec y = x,$ for $|x| \ge 1,\ 0 \le y \le \pi,$ and $y \ne \dfrac{\pi}{2}$.

Properties and Formulas:

- Derivatives of inverse trigonometric functions:

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}.$$
$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}.$$
$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}}.$$

- Integrals involving inverse trigonometric functions:

  For $a > 0$,

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C, \text{ and}$$
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C.$$

Summary:

In this lesson, we define the inverse trigonometric functions. We begin by looking closely at the sine function, observing that it is not one-to-one on its domain. Hence to define the inverse sine, we must restrict the domain of the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. On this domain, the sine function is one-to-one and has an inverse. More generally, none of the trigonometric functions are one-to-one; hence their domains must be restricted to define their inverses.

Example 1: Evaluating an Inverse Trigonometric Function

Evaluate $\arcsin\left(\frac{-1}{2}\right)$.

Solution:

The equation $\arcsin\left(\frac{-1}{2}\right) = y$ is equivalent to $\sin y = \frac{-1}{2}$, $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$. Hence $y = \arcsin\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$, because $\sin\left(\frac{-\pi}{6}\right) = \frac{-1}{2}$, and $\frac{-\pi}{6}$ lies in the interval.

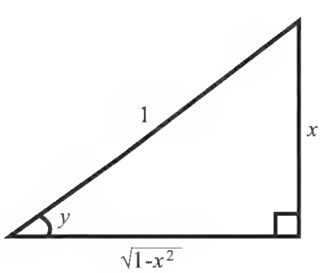
Sometimes you have to simplify complicated expressions involving trigonometric and inverse trigonometric functions. A right-triangle argument can be useful, as illustrated in the next example.

Example 2: Using Right Triangles

Given $y = \arcsin x$, $0 < y < \frac{\pi}{2}$, find $\cos y$.

Solution:

The given expression is equivalent to $\sin y = x = \frac{x}{1}$, so draw a right triangle, labeling one acute angle y, the opposite side x, and the hypotenuse 1. By the Pythagorean theorem, the third side is $\sqrt{1-x^2}$. Using the triangle, $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.



The derivatives of the inverse trigonometric functions are not trigonometric functions, but rather algebraic functions. The most important formulas are those for the inverse sine, inverse tangent, and inverse secant.

Example 3: Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left[\arcsin 2x\right] = \frac{1}{\sqrt{1-(2x)^2}}(2) = \frac{2}{\sqrt{1-4x^2}}$$
$$\frac{d}{dx}\left[\arctan 3x\right] = \frac{1}{1+(3x)^2}(3) = \frac{3}{1+9x^2}$$

The derivative formulas for the inverse trigonometric functions produce some interesting integral formulas.

Example 4: Integration with Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin\frac{x}{2} + C$$

Study Tips:

- The trigonometric functions (with domains appropriately restricted) are mirror images of the inverse trigonometric functions. Use your graphing utility to graph $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $y = \arctan x$ in the same viewing window to observe the symmetry across the line $y = x$.
- Recall that the graph of a one-to-one function satisfies the horizontal line test: A horizontal line will intersect the graph at most once. The trigonometric functions do not pass this test unless we restrict their domains.
- Calculus textbooks disagree on the definition of the inverse secant function. This stems from the fact that there are many ways to restrict the domain of the secant function to make it one-to-one. Using our definition, you can evaluate the arcsecant on a calculator using the formula $\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$.
- The arctangent function has 2 horizontal asymptotes, $y = \pm\frac{\pi}{2}$. These correspond to the vertical asymptotes $x = \pm\frac{\pi}{2}$ of the tangent function.

Pitfall:

- You must be very careful with the domains and ranges of the inverse trigonometric functions. For example, $\arccos\frac{1}{2} \neq \frac{-\pi}{3}$ despite the fact that $\cos\left(\frac{-\pi}{3}\right) = \frac{1}{2}$. The range of the arccosine function is $0 \le y \le \pi$, and $\arccos\frac{1}{2} = \frac{\pi}{3}$.

Problems:

1. Evaluate the expression $\arcsin 0$ without a calculator.

2. Evaluate the expression $\arccos 1$ without a calculator.

3. Use a right triangle to evaluate the expression $\sin\left(\arctan\frac{3}{4}\right)$.

4. Use a right triangle to evaluate the expression $\sec\left(\arcsin\dfrac{4}{5}\right)$.

5. Find the derivative of the function $f(x) = 2\arcsin(x-1)$.

6. Find the derivative of the function $f(x) = \arctan e^x$.

7. Find the derivative of the function $f(x) = \operatorname{arcsec} 2x$.

8. Find the equation of the tangent line to the graph of $y = \arctan\dfrac{x}{2}$ at the point $\left(2, \dfrac{\pi}{4}\right)$.

9. Find the indefinite integral: $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$.

10. Find the indefinite integral: $\displaystyle\int \frac{e^{2x}}{4+e^{4x}}\,dx$.

11. Evaluate the definite integral: $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$.

12. Derive the formula for the derivative of the inverse tangent function.

Lesson Twenty-Nine
Area of a Region between 2 Curves

Topics:

- Area of a region between 2 curves.
- Introduction to improper integrals.

Definitions and Theorems:

- If $f(x) \geq g(x)$ on the interval $[a,b]$, we say that the small rectangle of width Δx and height $f(x) - g(x)$ is a **representative rectangle**.
- An integral of the form $\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$ is called an **improper integral**.

Summary:

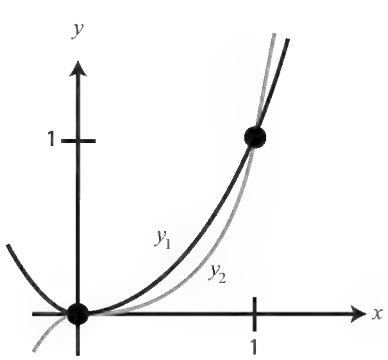
In this lesson, we revisit the area problem and discuss how to find the area of a region bounded by 2 curves. The approach is to imagine the region divided into representative rectangles of width Δx and height determined by the graphs of the functions. You then "add up" an infinite number of the representative rectangles, which corresponds to a definite integral.

Example 1: Finding the Area of a Region between 2 Curves

Find the area of the region in the first quadrant bounded by the graphs of the curves $y_1 = x^2$ and $y_2 = x^3$.

Solution:

By setting the equations equal to each other, you see that the curves intersect at $(0,0)$ and $(1,1)$. Sketch the region under consideration, noting that the curve $y_1 = x^2$ is above the other curve. Draw a representative rectangle on the region between the curves. The base of the rectangle has length Δx, and the height is $y_1 - y_2 = x^2 - x^3$.



Loosely speaking, the area between the curves is obtained by adding up the representative rectangles. That is, the area is the following definite integral:

Area = $\int_0^1 (y_1 - y_2)\,dx = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

Sometimes you encounter area problems of more complicated regions. For instance, the 2 curves might intersect at more than 2 points. In this case, you must find all the points of intersection and determine which curve is above the other on each interval determined by these points.

Example 2: Curves That Intersect at More than 2 Points

Find the area of the region between the graphs $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

Solution:

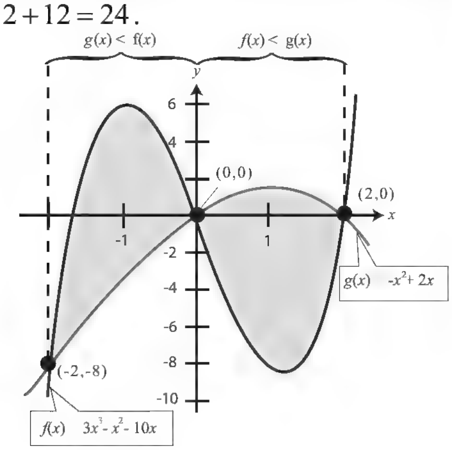
Set the equations equal to each other to find the points of intersection.

$$\begin{aligned} 3x^3 - x^2 - 10x &= -x^2 + 2x \\ 3x^3 - 12x &= 0 \\ 3x(x-2)(x+2) &= 0 \\ x &= 0,\ 2, -2 \end{aligned}$$

The curves intersect at 3 points: $(0,0)$, $(2,0)$, and $(-2,-8)$. On the interval $-2 \le x \le 0$, the graph of f is above that of g, whereas on the interval $0 \le x \le 2$, the graph of g is above that of f. Hence, the area is given by the 2 integrals shown below.

$$\text{Area} = \int_{-2}^{0} [f(x) - g(x)]dx \ + \ \int_{0}^{2} [g(x) - f(x)]dx$$

Although these integrals can be time consuming, they are easy to evaluate. We obtain Area $= 12 + 12 = 24$.



Note that if you had integrated $[f(x) - g(x)]$ on the entire interval $-2 \le x \le 2$, you would have obtained the incorrect answer of 0.

Sometimes by using limits, you can integrate a function defined on an unbounded interval. In the next example, we explore a so-called improper integral.

Example 3: The Area under an Infinite Region

The area under the curve $y = \dfrac{1}{x}$ from 1 to b, a constant greater than 1, is given by

$$\text{Area} = \int_{1}^{b} \frac{1}{x} dx \ = \ [\ln x]_1^b \ = \ \ln b - \ln 1 \ = \ \ln b.$$

Hence, as b tends to infinity, so does the area under this curve. On the other hand, the area under the curve $y=\frac{1}{x^2}$ from 1 to b is given by

$$\text{Area} = \int_1^b \frac{1}{x^2}\,dx = \int_1^b x^{-2}\,dx = \left[\frac{-1}{x}\right]_1^b = \frac{-1}{b}+1.$$

So as b tends to infinity, the area approaches 1. We say that the improper integral equals 1, and we write $\int_1^\infty \frac{1}{x^2}\,dx = 1$.

Study Tips:

- All of the applications of the integral calculus use the idea of adding up representative elements. For areas, we use representative rectangles. In the next lesson, we will use representative disks to find volumes.
- It is important to understand how to set up these applied problems. The actual integrations are usually uninteresting and time consuming.
- Sometimes it is more convenient to use horizontal representative rectangles and integrate with respect to the y variable. For example, the area bounded by $x=3-y^2$ and $x=y+1$ is given by

  $$\text{Area} = \int_{-2}^{1}\left[\left(3-y^2\right)-(y+1)\right]dy = \frac{9}{2}.$$

- Keep in mind that you can often simplify problems by taking advantage of symmetry. The area under the absolute value function from -5 to 5 is easy to calculate because the function is even:

  $$\text{Area} = \int_{-5}^{5}|x|\,dx = 2\int_0^5 x\,dx = 2\left[\frac{x^2}{2}\right]_0^5 = 25.$$

Pitfalls:

- You must be very careful to determine which function is above the other. For example, to find the area bounded by the curve $f(x)=x^2-6x$ and the x-axis, you should observe that the graph of f is below the axis. Hence, $A = \int_0^6\left[0-(x^2-6x)\right]dx = \int_0^6\left(6x-x^2\right)dx = 36$. If you had just integrated f from 0 to 6, you would have obtained a negative answer!
- Keep in mind that although a definite integral can be positive, negative, or zero, the area of a region is always nonnegative.

Problems:

1. Find the area of the region bounded by the curves $y=x^2-1$, $y=-x+2$, $x=0$, and $x=1$.

2. Find the area of the region bounded by the curves $y=-x^3+3$, $y=x$, $x=-1$, and $x=1$.

3. Find the area of the region bounded by the curves $y=\sqrt[3]{x-1}$ and $y=x-1$.

4. Find the area of the region bounded by the curves $x=y^2$ and $x=y+2$.

5. Find the area of the region bounded by the curves $y=\cos x$ and $y=2-\cos x$, $0\le x\le 2\pi$.

6. Find the area of the region bounded by the curves $y=xe^{-x^2}$ and $y=0$, $0\le x\le 1$.

7. Find the area of the 2 regions bounded by the curves $y_1 = 3(x^3 - x)$ and $y_2 = 0$.

8. Find the area of the 2 regions bounded by the curves $y_1 = (x-1)^3$ and $y_2 = x - 1$.

9. Consider the area bounded by $x = 4 - y^2$ and $x = y - 2$. Calculate this area by integrating with respect to x, and then with respect to y. Which method is simpler?

10. Set up and evaluate the definite integral that gives the area of the region bounded by the graph of $f(x) = x^3$ and its tangent line at $(1,1)$.

Lesson Thirty
Volume—The Disk Method

Topics:

- The disk method for finding the volume of a solid of revolution.
- The washer method for finding the volume of a solid of revolution.

Definitions and Theorems:

- If a region in the plane is revolved about a line, the resulting solid is a **solid of revolution**, and the line is called the **axis of revolution**.
- The representative element for the disk method is a small disk of width Δx and area πR^2.

Summary:

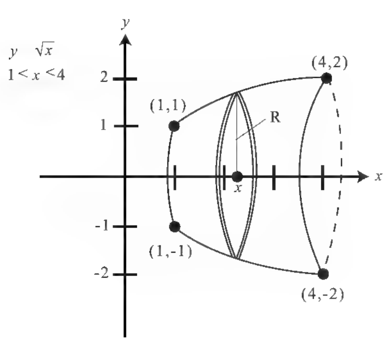
In this lesson, we learn how to calculate the volume of a solid of revolution. As in the area problem, we imagine the solid divided into small disks of volume $\Delta V = \pi R^2 \Delta x$. The radius R is usually determined by the function f. You then "add up" these disks with a definite integral.

Example 1: Using the Disk Method

Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x}$ and $y = 0,\ 1 \le x \le 4,$ about the x-axis.

Solution:

The volume of a representative disk is $\pi R^2 \Delta x = \pi\left[\sqrt{x}\right]^2 \Delta x = \pi x \Delta x$.



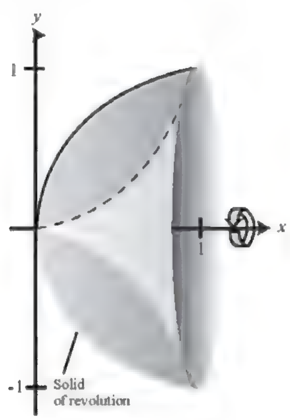
The volume of the solid of revolution is obtained by adding up the representative disks. That is, the volume is the following definite integral:

$$\text{Volume} = \int_1^4 \pi R^2\,dx = \int_1^4 \pi x\,dx = \pi\left[\frac{x^2}{2}\right]_1^4 = \pi\left(8 - \frac{1}{2}\right) = \frac{15}{2}\pi.$$

You can find volumes of solids of revolutions with holes by using the washer method. That is, the representative element is a washer formed by revolving a small rectangle about an axis. If r and R are the inner and outer radii of the washer, and the thickness is Δx, then the volume of the washer is $\Delta V = \pi(R^2 - r^2)\Delta x$.

Example 2: Using the Washer Method

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.



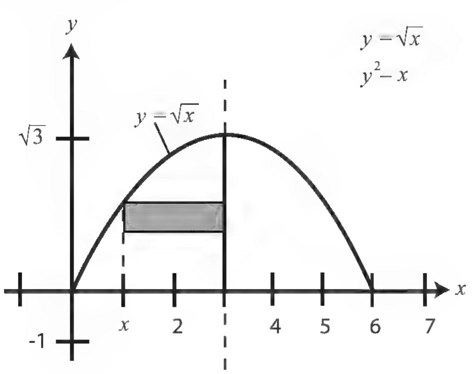
Solution:

From the figure you see that the outer and inner radii are $R(x) = \sqrt{x}$ and $r(x) = x^2$. The volume is given by the integral $V = \pi\int_0^1\left[R(x)^2 - r(x)^2\right]dx = \pi\int_0^1\left[\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right]dx = \pi\int_0^1\left[x - x^4\right]dx = \dfrac{3\pi}{10}$.

Sometimes the axis of rotation is not the usual x- or y-axis. The underlying principles are the same, except the radius function might be more difficult to determine.

Example 3: Revolving about a Line That Is Not a Coordinate Axis

Find the volume of the solid formed by revolving the region bounded the graphs of $f(x) = \sqrt{x}$, $y = 0$, and $x = 3$ about the line $x = 3$.



Solution:

Since the axis of rotation is vertical, we will integrate with respect to the variable y. So rewrite the equation $y = \sqrt{x}$ as $x = y^2$. The radius function is $3 - x = 3 - y^2$, and the limits of integration are $0 \le y \le \sqrt{3}$. The volume integral becomes $V = \pi\int_0^{\sqrt{3}}\left(3 - y^2\right)^2 dy = \dfrac{24\pi\sqrt{3}}{5}$.

Study Tips:

- Notice how you calculate volumes by using representative elements, either disks or washers. There is no need to memorize formulas; rather, you just need to understand the underlying principle of adding up representative elements.
- The integrals for volumes can be difficult. For example, if we rotate 1 arch of the sine curve about the x-axis, we obtain the following integral for the resulting volume: $V = \pi\int_0^{\pi} \sin^2 x\,dx$. We will learn how to find an antiderivative for $\sin^2 x$ in Lesson Thirty-Four.
- Many volume problems give rise to integrals involving even functions, which can simplify the computations.

Pitfall:

- Take extra care in setting up volume integrals. In particular, it is difficult to determine the radius function if the axis is not a coordinate axis. I suggest that you check your radius function by substituting in various values and seeing if the result makes sense. For instance, in Example 3 above, if $y = 0$, then the radius is 3, whereas if $y = \sqrt{3}$, the radius is 0.

Problems:

1. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations $y = -x + 1$, $y = 0$, and $x = 0$ about the x-axis.

2. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations $y = 4 - x^2$, $y = 0$, and $x = 0$ about the x-axis.

3. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations $y = 2$ and $y = 4 - \dfrac{x^2}{4}$ about the x-axis.

4. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the graphs of the equations $y = x^2$, $y = 4$, and $x = 0$ about the y-axis.

5. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the graph of the equation $y = \sqrt{16 - x^2}$ about the y-axis.

6. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the graphs of the equations $y = x^{2/3}$, $y = 1$, and $x = 0$ about the y-axis.

7. Find the volume of the solid generated by revolving the region bounded by the curves $y = \sec x$ and $y = 0$, $0 \le x \le \dfrac{\pi}{3}$, about the line $y = 4$.

8. Find the volume of the solid generated by revolving the region bounded by the curves $y = 5 - x$, $y = 0$, $y = 4$, and $x = 0$ about the line $x = 5$.

9. Use the disk method to verify that the volume of a sphere is $\dfrac{4}{3}\pi r^3$.

10. A sphere of radius r is cut by a plane h $(h < r)$ units above the equator. Find the volume of the solid (spherical segment) above the plane.

11. Describe the volume represented by the definite integral: $\pi\int_2^4 y^4\,dy$.

Lesson Thirty-One
Volume—The Shell Method

Topics:

- The shell method for finding the volume of a solid of revolution.
- The advantages and disadvantages of the disk and shell methods.

Definitions and Theorems:

- The representative element for the shell method (with vertical axis of rotation) is a small cylindrical shell of thickness Δx, height $h(x)$, and radius $r(x)$.
- A **torus** is formed by revolving the region bounded by the circle $x^2 + y^2 = r^2$ about the line $x = R$ $(r < R)$.

Summary:

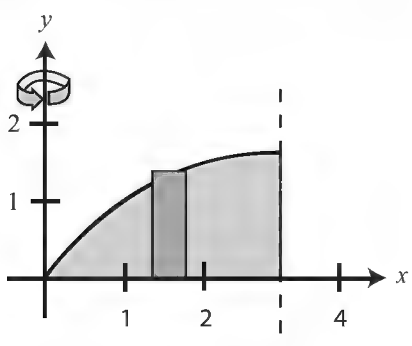
In this lesson, we continue our study of volumes of solids of revolution. For a vertical axis of rotation, we imagine the solid divided into small cylindrical shells of volume $\Delta V = 2\pi p(x)h(x)\Delta x$. Here, $p(x)$ is the radius, and $2\pi p(x)$ represents the circumference of the shell. In many cases, the radius $p(x)$ is simply x, and the height is determined by the function f. You then "add up" these shells to obtain the corresponding definite integral.

Example 1: Using the Shell Method

Find the volume of the solid formed by revolving the region in the first quadrant bounded by $y = \sqrt{x}$, $0 \le x \le 3$, about the y-axis.

Solution:

The volume of a representative shell is $2\pi p(x)h(x)\Delta x$, which equals $2\pi x f(x)\Delta x$.



The volume of the solid of revolution is obtained by adding up the representative shells. That is, the volume is the following definite integral:

$$V = \int_0^3 2\pi x f(x)\,dx = 2\pi\int_0^3 x\sqrt{x}\,dx = 2\pi\int_0^3 x^{\frac{3}{2}}\,dx = 2\pi\frac{2}{5}\left[x^{\frac{5}{2}}\right]_0^3 = \frac{4\pi}{5}\sqrt{243} = \frac{36\sqrt{3}\,\pi}{5}.$$

We now know how to calculate volumes of solids of revolution using both the disk method and the shell method. In many problems, one method will turn out to be easier than the other.

Example 2: Using the Disk Method and the Shell Method

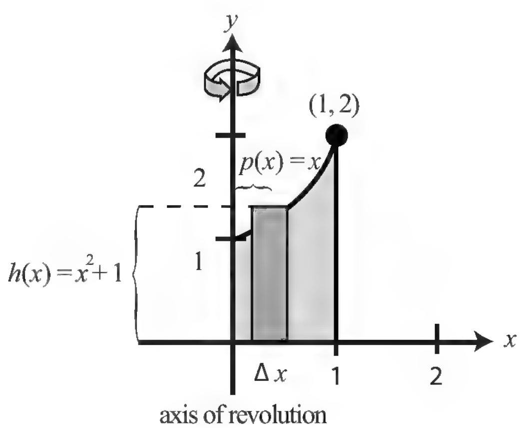
The region in the first quadrant bounded by the graphs of $y = x^2 + 1$ and $x = 1$ is revolved about the y-axis. Set up the integrals for this volume using the shell method and the disk method.

Solution:

From the figure, you see that the shell method gives the volume as a single integral:

$$V = 2\pi\int_0^1 p(x)h(x)\,dx = 2\pi\int_0^1 x(x^2+1)\,dx.$$

This integral is easy to calculate, and the final answer is $\dfrac{3\pi}{2}$.



On the other hand, the disk method requires 2 integrals. The first represents the volume of a right circular cylinder. Notice that we are now integrating with respect to the variable y, so we have to express x as a function of y: $x = \sqrt{y-1}$.

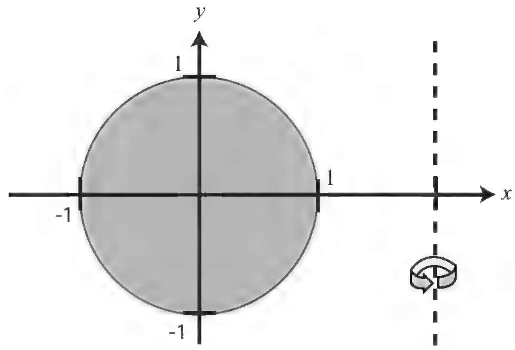
$$V = \pi\int_0^1 (1^2 - 0^2)dy + \pi\int_1^2 \left(1^2 - \left(\sqrt{y-1}\right)^2\right)dy$$

These 2 integrals yield $V = \pi + \dfrac{\pi}{2}$, and we obtain the same answer as before, $\dfrac{3\pi}{2}$. But it is clear that the shell method is easier.

Example 3: The Volume of a Torus

The circle $x^2 + y^2 = 1$ is revolved about the vertical line $x = 2$, forming a doughnut-shaped solid called a torus. Set up the integral for the volume of this torus.

Solution:



We will calculate the portion of the torus above the x-axis and double the answer. By doing this, we can solve for y in the equation for the circle to obtain the height function $y = \sqrt{1-x^2}$. From the figure, you see that the radius is $p(x) = 2 - x$ and the limits of integration are $x = -1$ to $x = 1$. Hence the volume is given by

$$V = 2\int_{-1}^{1} 2\pi(2-x)\sqrt{1-x^2}\, dx \, .$$

Study Tips:

- If the axis of rotation is horizontal, the volume of the representative element is $\Delta V = 2\pi p(y)h(y)\Delta y$.
- By splitting up the integral in Example 3 above and using some geometry, you can actually calculate the volume of the torus:
  $V \;=\; 2\int_{-1}^{1} 2\pi(2-x)\sqrt{1-x^2}\, dx \;=\; 8\pi\int_{-1}^{1} \sqrt{1-x^2}\, dx \;-\; 4\pi\int_{-1}^{1} x\sqrt{1-x^2}\, dx$.
  The first integral is the area of a semicircle, and the second integral is 0 because the integrand is an odd function. Hence the volume is $8\pi\left(\dfrac{\pi}{2}\right) = 4\pi^2$.

Pitfall:

- You will see volume problems with different axes of rotation. You should take extra care in setting up the resulting integrals, especially the radius function p. For example, in the torus example above, the radius is $p(x) = 2 - x$.

Problems:

1. The region in the first quadrant bounded by the graphs of the equations $y = x$ and $x = 2$ is revolved about the y-axis. Use the shell method to find the volume of the resulting solid.
2. The region in the first quadrant bounded by the graph of the equation $y = 1 - x$ is revolved about the y-axis. Use the shell method to find the volume of the resulting solid.
3. The region in the first quadrant bounded by the graphs of the equations $y = x^2$ and $x = 3$ is revolved about the y-axis. Use the shell method to find the volume of the resulting solid.
4. The region in the first quadrant bounded by the graphs of the equations $y = x$ and $x = 2$ is revolved about the x-axis. Use the shell method to find the volume of the resulting solid.

5. The region in the first quadrant bounded by the graphs of the equations $y = \frac{1}{x}$, $x = 1$, and $x = 2$ is revolved about the x-axis. Use the shell method to find the volume of the resulting solid.
6. The region in the first quadrant bounded by the graph of the equation $y = 4x - x^2$ is revolved about the line $x = 5$. Use the shell method to find the volume of the resulting solid.
7. Find the volume of the torus formed by revolving the region bounded by the circle $x^2 + y^2 = r^2$ about the line $x = R$ $(r < R)$.
8. Describe the volume represented by the definite integral: $2\pi\int_0^2 x^3\,dx$.
9. Describe the volume represented by the definite integral: $2\pi\int_0^1 \left(y - y^{\frac{3}{2}}\right)dy$.

Lesson Thirty-Two
Applications—Arc Length and Surface Area

Topics:

- The arc length of a curve.
- The area of a surface of revolution.

Definitions and Theorems:

- We say that the graph of a function f is **smooth** on the interval $[a,b]$ if f' is continuous on that interval.
- The representative element for the arc length of a smooth curve is a small piece of curve of length $\Delta s \approx \sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$.
- The **differential of arc length** is $ds = \sqrt{1+f'(x)^2}\,dx$.

Formulas:

- Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a,b]$. The **arc length** of f between a and b is $s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$.
- Let $y = f(x)$ have a continuous derivative on the interval $[a,b]$. The area of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is $S = 2\pi\int_a^b r(x)\sqrt{1+[f'(x)]^2}\,dx$, where $r(x)$ is the distance between the graph of f and the axis of revolution.
- Hanging cables (and the St. Louis Arch) are in the shape of a **catenary**. The formula for a typical catenary is $y = 75\left(e^{\frac{x}{150}}+e^{-\frac{x}{150}}\right),\ -100 \le x \le 100$.

Summary:

In this lesson, we look at 2 more applications of integration: arc length and surface area. The formula for the arc length of a curve is based on a representative element consisting of a small piece of arc whose length is determined by the Pythagorean theorem: $\Delta s \approx \sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$.

Example 1: The Length of a Line Segment

Find the length of the line segment from $(0,0)$ to $(3,4)$.

Solution:

You already know that the length of the segment can be calculated by the distance formula: $s=\sqrt{3^2+4^2}=\sqrt{25}=5$. Let's verify this with the arc length formula. The equation of the line segment is $y=\frac{4}{3}x,\ 0\le x\le 3$. Because $y'=\frac{4}{3}$, the length of the line segment is $s\ =\ \int_a^b\sqrt{1+f'(x)^2}dx\ =$ $\int_0^3\sqrt{1+\left(\frac{4}{3}\right)^2}\,dx\ =\ \int_0^3\sqrt{\left(\frac{3^2+4^2}{3^2}\right)}dx\ =\ \int_0^3\frac{5}{3}dx\ =\ \left[\frac{5}{3}x\right]_0^3\ =\ 5$.

The area of a surface of revolution is based on the arc length formula. The idea is to imagine a piece of arc $ds=\sqrt{1+\left[f'(x)\right]^2}\,dx$ rotated about an axis.

Example 2: The Area of a Surface of Revolution

Set up the integral for the area of the surface formed by revolving the graph of $y=x^3,\ 0\le x\le 1$, about the x-axis.

Solution:

In this case, the radius is $r(x)=x^3$. Since $f'(x)=3x^2$, we have

$$S\ =\ 2\pi\int_0^1 x^3\sqrt{1+\left(3x^2\right)^2}\,dx\ =\ 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx.$$

This integral can be calculated using substitution $(u=x^4)$, and the answer is $\frac{\pi}{27}\left(10^{\frac{3}{2}}-1\right)\approx 3.563$.

Study Tips:

- The integrals that result from using the arc length formula or surface area formula can be very difficult to evaluate. In many applications, approximate integration, such as the trapezoidal rule, is the only way to calculate the integral.
- In Example 2 above, if we had revolved the curve about the y-axis, then the radius function would be $r(x)=x$, and the surface area integral would be $S\ =\ 2\pi\int_0^1 x\sqrt{1+\left(3x^2\right)^2}\,dx\ =\ 2\pi\int_0^1 x\sqrt{1+9x^4}\,dx$.
- There are many other applications of the definite integral. They all are based on the idea of a representative element being added up. Rather than memorize a lot of formulas, keep this underlying theme in mind when you approach a new problem.

Pitfall:

- Be careful to distinguish between exact answers and approximations. In Example 2 above, the exact answer is $\frac{\pi}{27}\left(10^{\frac{3}{2}}-1\right)$, whereas 3.563 is an approximation. Do not write $S=3.563$, but rather $S\approx 3.563$.

Problems:

1. Find the distance between the points $(0,0)$ and $(8,15)$ using the distance formula and the arc length formula.
2. Find the arc length of the graph of the function $y=\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$ over the interval $0\le x\le 1$.
3. Find the arc length of the graph of the function $y=\frac{2}{3}x^{\frac{3}{2}}+1$ over the interval $0\le x\le 1$.
4. Find the arc length of the graph of the function $y=\frac{x^4}{8}+\frac{1}{4x^2}$ over the interval $1\le x\le 3$.
5. Find the arc length of the graph of the function $y=\frac{1}{2}\left(e^x+e^{-x}\right)$ over the interval $0\le x\le 2$.
6. Set up the integral for the arc length of the graph of the function $y=\sin x$ over the interval $0\le x\le \pi$. Use a graphing utility to approximate this integral.
7. Set up and evaluate the definite integral for the area of the surface generated by revolving $y=\frac{x}{2},\ 0\le x\le 6,$ about the x-axis.
8. Set up and evaluate the definite integral for the area of the surface generated by revolving $y=\sqrt{4-x^2},\ -1\le x\le 1$, about the x-axis.
9. Set up and evaluate the definite integral for the area of the surface generated by revolving $y=9-x^2,\ 0\le x\le 3$, about the y-axis.
10. Set up and evaluate the definite integral for the area of the surface generated by revolving $y=2x+5,\ 1\le x\le 4$, about the y-axis.

Lesson Thirty-Three
Basic Integration Rules

Topic:

- Basic integration rules.

Formula:

- A summary of all the important integration formulas is given in the Formulas appendix.

Summary:

In this lesson, we review all the integration formulas we have studied so far in the course and show how to apply them in various examples. As you have noticed, finding antiderivatives is more difficult than calculating derivatives. You will see in the following example how similar integrands can have very different antiderivatives.

Example 1: Two Similar Integrals

Calculate the integrals:

1. $\int \frac{4}{x^2+9}dx$.
2. $\int \frac{4x}{x^2+9}dx$.

Solution:

1. For this integral you use the arctangent rule:

$$\int \frac{4}{x^2+9}dx = 4\int \frac{1}{x^2+3^2}dx = 4\left(\frac{1}{3}\arctan\frac{x}{3}\right)+C = \frac{4}{3}\arctan\frac{x}{3}+C.$$

2. This time you should use integration by substitution, with $u=x^2$, $du=2x\,dx$:

$$\int \frac{4x}{x^2+9}dx = 2\int \frac{1}{x^2+9}(2x)dx = 2\ln\left|x^2+9\right|+C = 2\ln\left(x^2+9\right)+C = \ln\left(x^2+9\right)^2+C.$$

Notice the use of logarithm properties to simplify the answer for the second integral.

Integration problems can require cleverness and creativity. The following example illustrates a standard technique: multiplication and division by a special factor.

Example 2: Finding an Antiderivative

Calculate $\int \frac{1}{1-\sin x}dx$.

Solution:

It is not obvious how to proceed. The trick here is to multiply and divide by $1+\sin x$, as follows:

$$\int \frac{1}{1-\sin x}dx = \int \frac{1}{1-\sin x}\frac{1+\sin x}{1+\sin x}dx = \int \frac{1+\sin x}{1-\sin^2 x}dx .$$

We use the fundamental identity $\sin^2 x+\cos^2 x=1$ to simplify the denominator to $\cos^2 x$. Thus we can split up the integral into 2 pieces.

$$\int \frac{1}{1-\sin x}dx = \int \frac{1+\sin x}{\cos^2 x}dx = \int \frac{1}{\cos^2 x}dx + \int \frac{\sin x}{\cos^2 x}dx$$

$$= \int \sec^2 x\,dx + \int \sec x \tan x\,dx = \tan x + \sec x + C$$

This answer can also be written as $\dfrac{-\cos x}{\sin x-1}+C_1$. You are asked to show that both answers are correct in the problems below.

Study Tips:

- The key to becoming adept at finding antiderivatives is practice, practice, practice! And remember that you can check your answers by differentiation.
- Sometimes you will need long division to simplify an integral involving rational functions. For instance, notice how long division allows us to rewrite a complicated integrand into 2 easier ones.

$$\int \frac{4x^2}{x^2+9}dx = \int \left(4-\frac{36}{x^2+9}\right)dx$$

Pitfall:

- A common error with fractions is splitting up the denominator rather than the numerator.

$$\frac{1}{1-\sin x} \neq \frac{1}{1}-\frac{1}{\sin x}$$

Problems:

1. Evaluate the integral: $\displaystyle\int \frac{9}{(t-8)^2}dt$.

2. Evaluate the integral: $\displaystyle\int \frac{t^2-3}{-t^3+9t+1}dt$.

3. Evaluate the integral: $\displaystyle\int \frac{e^x}{1+e^x}dx$.

4. Evaluate the integral: $\displaystyle\int \frac{x^2}{x-1}dx$.

5. Evaluate the integral: $\displaystyle\int \frac{1+\sin x}{\cos x}dx$.

6. Evaluate the integral: $\displaystyle\int \frac{1}{\cos\theta-1}d\theta$.

7. Evaluate the integral: $\int \frac{6}{\sqrt{10x-x^2}}\,dx$.

8. Solve the differential equation $\left(4+\tan^2 x\right)y' = \sec^2 x$.

9. Evaluate the definite integral: $\int_0^8 \frac{2x}{\sqrt{x^2+36}}\,dx$.

10. Evaluate the definite integral: $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\,dx$.

11. Explain why the antiderivative $y_1 = e^{x+C_1}$ is equivalent to the antiderivative $y_2 = Ce^x$.

12. Find the arc length of the graph of $y = \ln(\sin x)$ from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$.

13. Show that the 2 answers to Example 2 are equivalent. That is, show that $\tan x + \sec x + C = \frac{-\cos x}{\sin x - 1} + C_1$.

Lesson Thirty-Four
Other Techniques of Integration

Topics:

- Integration by parts.
- Trigonometric integrals.

Formula:

- Integration by parts: $\int u\,dv = uv - \int v\,du$.

Summary:

We begin this lesson by looking at a powerful method for finding antiderivatives: integration by parts. This technique is based on the product rule for derivatives. It is convenient to use differential notation, although this is not necessary. Given an integral, you must make choices for u and dv. You then differentiate u, integrate dv, and substitute these values into the formula for integration by parts. The original integral will be converted to another simpler integral.

Example 1: Integration by Parts

Calculate the integral: $\int xe^x\,dx$.

Solution:

Let's try the choices $u = x$ and $dv = e^x\,dx$. Differentiating u, we have $du = dx$. And integrating dv, we obtain $v = e^x$. So the original integral becomes

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

You do not need to add a constant of integration when you integrate dv; any antiderivative will do. Note also that the formula for integration by parts converted the original (difficult) integral to an easier one.

Example 2: Solving a Differential Equation Using Integration by Parts

Solve the differential equation $y' = \ln x$.

Solution:

We are trying to evaluate the integral $y = \int \ln x\,dx$. Setting $u = \ln x$ and $dv = dx$ gives $du = \frac{1}{x}$ and $v = x$. Now use integration by parts:

$$\int u\,dv = \int \ln x\,dx = uv - \int v\,du = \ln x(x) - \int x\frac{1}{x}dx = x\ln x - \int 1\,dx = x\ln x - x + C.$$

Applications of calculus can often lead to integrals that require integration by parts. In the following example, a volume is calculated using the shell method.

Example 3: An Application to Volumes

The region bounded by $y = \ln x$, $y = 0$, and $x = e$ is revolved about the y-axis. Find the resulting volume.

Solution:

Using the shell method, $V = 2\pi\int_1^e x\ln x\,dx$. Now use integration by parts.

$$u = \ln x,\; dv = x\,dx,\; du = \frac{1}{x}dx, \text{ and } v = \frac{x^2}{2}.$$

$$\int x\ln x\,dx = uv - \int v\,du = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$

$$V = 2\pi\int_1^e x\ln x\,dx = 2\pi\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^e = 2\pi\left[\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}\right] = \frac{e^2+1}{2}\pi.$$

Integrals involving products of trigonometric functions occur in many applications. Let's look at an example involving sine and cosine.

Example 4: A Trigonometric Integral

Determine the following antiderivatives:

a. $\int \sin^5 x\cos x\,dx$.

b. $\int \sin^4 x\cos^3 x\,dx$.

Solution:

a. This problem is simply substitution, where $u = \sin x$ and $du = \cos x\,dx$. Hence

$$\int \sin^5 x\cos x\,dx = \frac{\sin^6 x}{6} + C.$$

b. Since we have an odd power of cosine, we save 1 factor and convert $\cos^2 x$ to $(1 - \sin^2 x)$. Hence

$$\begin{aligned}\int \sin^4 x\cos^3 x\,dx &= \int \sin^4 x\cos^2 x\cos x\,dx\\ &= \int \sin^4 x(1-\sin^2 x)\cos x\,dx\\ &= \int (\sin^4 x - \sin^6 x)\cos x\,dx\\ &= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{x} + C.\end{aligned}$$

Study Tips:

- For integration by parts, you must guess the values of u and dv. One suggestion is to choose dv to be something that fits a basic integration rule. Another suggestion is to let u be the portion of the integrand whose derivative is a function simpler than u.
- Sometimes you might have to do integration by parts more than once to find an antiderivative.
- Some students find it convenient to memorize the integral formula for the natural logarithm function: $\int \ln x\,dx = x\ln x - x + C$. You can verify that this is correct by differentiating the answer.
- The key to becoming adept at finding antiderivatives is practice, practice, practice! And remember that you can check your answers by differentiation.

- If an integrand consists of a product of sines and cosines, $\int \sin^n x \cos^m x\,dx$, then you can use the technique of Example 3 above if either power is odd. If both powers are even, the antiderivative is more difficult to calculate. Similarly, if an integrand consists of a product of tangents and secants, $\int \tan^n x \sec^m x\,dx$, then the same technique works if either n is odd or m is even.

Problems:

1. Evaluate $\int x \sin 3x\,dx$ using integration by parts with $u = x$ and $dv = \sin 3x\,dx$.

2. Evaluate $\int x^3 \ln x\,dx$ using integration by parts with $u = \ln x$ and $dv = x^3\,dx$.

3. Evaluate the integral: $\int xe^{-4x}\,dx$.

4. Evaluate the integral: $\int x \cos x\,dx$.

5. Evaluate the integral: $\int \arctan x\,dx$.

6. Evaluate the integral: $\int \frac{\ln 2x}{x^2}\,dx$.

7. Use substitution and integration by parts to evaluate the integral: $\int \sin\sqrt{x}\,dx$.

8. Evaluate the integral: $\int \sin^7 2x \cos 2x\,dx$.

9. Evaluate the integral: $\int \sin^3 x \cos^2 x\,dx$.

10. Evaluate the integral: $\int \sec^2 x \tan x\,dx$.

11. Evaluate the integral: $\int \tan^2 x \sec^4 x\,dx$.

12. Evaluate the integral: $\int \cos^2 x\,dx$.

13. Which integral is easier to evaluate, $\int \tan^{400} x \sec^2 x\,dx$ or $\int \tan^{400} x \sec x\,dx$? Why?

Lesson Thirty-Five
Differential Equations and Slope Fields

Topics:

- Differential equations.
- Solution curves.
- Particular solutions to differential equations.
- Slope fields.
- Separation of variables.

Definitions and Theorems:

- A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. The order of a differential equation is determined by the highest-order derivative in the equation.
- A function $y = f(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$.
- A particular solution of a differential equation is obtained from initial conditions that give the value of the dependent variable or one of its derivatives for particular values of the independent variable.
- The general solution of a first-order differential equation represents a family of curves known as **solution curves**, one for each value of the arbitrary constant.
- Given a differential equation in x and y, $y' = F(x, y)$, the equation determines the derivative at each point (x, y). If you draw short line segments with slope $F(x, y)$ at selected points (x, y), then these line segments form a **slope field**.

Summary:

Our final 2 lessons are devoted to differential equations. In this lesson, we look at solutions to differential equations, slope fields, and a solution technique called separation of variables. Our final lesson will examine some real-life applications of differential equations.

Example 1: Verifying a Solution to a Differential Equation

Given the differential equation $y' + 2y = 0$, verify that $y = Ce^{-2x}$ is a solution for any constant C.

Solution:

The derivative of the proposed solution is $y' = -2Ce^{-2x}$. Substituting into the original differential equation, we have $y' + 2y = -2Ce^{-2x} + 2(Ce^{-2x}) = 0$. Notice that for each value of C, we obtain a specific solution curve of the differential equation.

Given a differential equation, it is possible to visualize the solution without actually solving the equation. A slope field gives the slope of the solution at any point (x, y) in the domain of the equation.

Example 2: Sketching a Slope Field

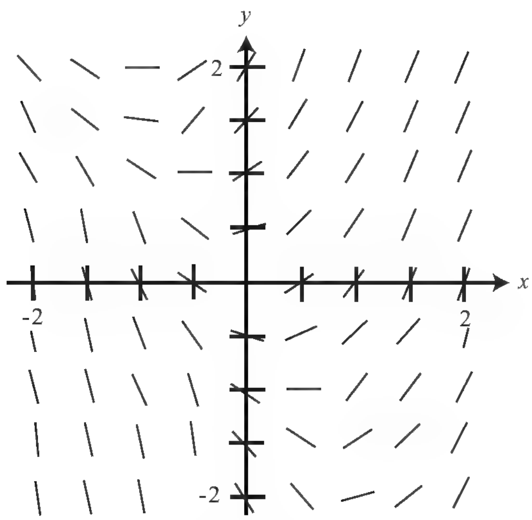
Sketch a slope field for the differential equation $y' = 2x + y$.

Solution:

We begin by making a table showing the slopes at several points. The table shown is a small sample. The slopes at many other points should be calculated to get a representative slope field.

x	-2	-2	-1	-1	0	0	1	1	2
y	-1	1	-1	1	-1	1	-1	1	-1
$2x+y$	-5	-3	-3	-1	-1	1	1	3	3

You can then draw small line segments at the points with their respective slopes, as shown in the figure below.



There are many techniques for solving differential equations. One of the simplest is separation of variables, in which you attempt to move all terms involving x to one side of the equation and all terms with y to the other side.

Example 3: Separation of Variables

Solve the differential equation $\frac{dy}{dx} = \frac{y}{x^2}$, $y(1) = 3$.

Solution:

Move dx to the right-hand side and y to the left-hand side. Then integrate both sides simultaneously.

$$\frac{1}{y}dy = \frac{1}{x^2}dx \Rightarrow \int \frac{1}{y}dy = \int \frac{1}{x^2}dx \Rightarrow \ln|y| = \frac{-1}{x} + C_1$$

$$y = e^{\left(-\frac{1}{x}\right)+C_1} = Ce^{-\frac{1}{x}}$$

We now use the initial condition $y(1) = 3$ to determine the constant C.

$$y(1) = 3 = Ce^{\frac{-1}{1}} = Ce^{-1}$$
$$C = 3e$$

The final answer is $y = (3e)e^{-\frac{1}{x}} = 3e^{\frac{(x-1)}{x}}$, $x > 0$. This function satisfies the differential equation and the initial condition.

Study Tips:

- The differential equation $s''(t) = -32$ is a second-order equation representing the acceleration of a falling body under the influence of gravity. Its general solution has 2 arbitrary constants: $s(t) = -16t^2 + C_1 t + C_2$.
- Slope fields can be tedious to draw by hand. Most graphing utilities have built-in programs for slope fields.
- In separation of variables, you do not need to write 2 constants of integration when you integrate both sides of the equation. The constants can be combined into 1 constant of integration.
- Notice the difficult precalculus manipulations in Example 3 above. In particular, the expression $y = e^{-\frac{1}{x}+C_1} = e^{-\frac{1}{x}} e^{C_1}$ is equivalent to $y = Ce^{-\frac{1}{x}}$ because e^{C_1} is a constant.

Problems:

1. Verify that $y = Ce^{4x}$ is a solution to the differential equation $y' = 4y$.
2. Verify that $y = C_1 \sin x + C_2 \cos x$ is a solution to the differential equation $y'' + y = 0$.
3. Verify that $y = 4e^{-6x^2}$ is a solution to the differential equation $y' = -12xy,\ y(0) = 4$.
4. Use integration to find a general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x}{1+x^2}$.
5. Use integration to find a general solution of the differential equation $\dfrac{dy}{dx} = x\sqrt{x-6}$.
6. Sketch a slope field for the differential equation $y' = y - 4x$.
7. Use separation of variables to solve the differential equation $\dfrac{dy}{dx} = x + 3$.
8. Use separation of variables to solve the differential equation $\dfrac{dy}{dx} = \dfrac{5x}{y}$.
9. Use separation of variables to solve the differential equation $\dfrac{dy}{dx} = x(1+y)$.
10. Use separation of variables to solve the differential equation $\dfrac{dy}{dt} = ky$, where y is a function of t and k is a constant.

Lesson Thirty-Six
Applications of Differential Equations

Topics:

- Growth and decay models in differential equations.
- Radioactive decay.
- Population growth.
- Newton's law of cooling.

Definitions and Theorems:

- The solution to the **growth and decay model** $\frac{dy}{dt} = ky$ is $y = Ce^{kt}$. In this model, C is the initial value and k is the constant of proportionality. You have exponential growth if $k > 0$ and exponential decay if $k < 0$.
- **Newton's law of cooling** states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

Summary:

In our final lesson, we look closely at 3 real-life applications of differential equations: radioactive decay, population growth, and Newton's law of cooling. The general principles illustrated in Example 1 apply to all 3 models.

Example 1: Radioactive Decay

Suppose that 10 grams of the plutonium isotope Pu-239 were released in the Chernobyl nuclear accident. How long would it take for the 10 grams to decay to 1 gram? The half-life of Pu-239 is 24,100 years.

Solution:

Let y represent the mass in grams of the plutonium. Because the rate of decay is proportional to y, the model is $y = Ce^{kt}$. Since $y = 10$ when $t = 0$, you have $C = 10$, and the model becomes $y = 10e^{kt}$.

The half-life information—that $y = 5$ when $t = 24{,}100$—will allow us to find the constant of proportionality, k.

$$\begin{aligned} 5 &= 10e^{k(24{,}100)} \\ \frac{1}{2} &= e^{24{,}100k} \\ \ln\frac{1}{2} &= 24{,}100k \\ k &= \frac{1}{24{,}100}\ln\frac{1}{2} \approx -0.000028761 \end{aligned}$$

Our model is now $y = 10e^{-0.000028761t}$. Notice that k is negative, indicating exponential decay. The final step is to find the time t when y has decayed to 1 gram.

$$1 = 10e^{-0.000028761t}$$
$$\ln\frac{1}{10} = -0.000028761t$$
$$t \approx 80{,}059$$

Thus it takes approximately 80,000 years for the plutonium to decay to 1 gram.

Study Tips:

- You will often need to use the inverse properties of logarithms and exponentials. For example, in the example above, we used the inverse properties to show that $\frac{1}{2} = e^{24{,}100k} \Rightarrow \ln\frac{1}{2} = \ln e^{24{,}100k} = 24{,}100k$.
- Applications of calculus and differential equations can be very complicated. The examples we've seen in this course are fairly simple but illustrate the core principles of how to apply calculus to real-life models.

Pitfall:

- Don't forget that $\ln x < 0$ for $0 < x < 1$. In particular, $\ln\frac{1}{2} = \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2 < 0$.

Problems:

1. The rate of change of y is proportional to y. When $x = 0$, $y = 6$, and when $x = 4$, $y = 15$. Write and solve the differential equation that models this verbal statement. Then find the value of y when $x = 8$.
2. The rate of change of N is proportional to N. When $t = 0$, $N = 250$, and when $t = 1$, $N = 400$. Write and solve the differential equation that models this verbal statement. Then find the value of N when $t = 4$.
3. Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?
4. The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours. Find the initial population. Also, how many bacteria will there be after 8 hours?
5. Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after another 2 months?
6. When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F, its core temperature is 1500°F. One hour after it is removed, the core temperature is 1120°F. Find the core temperature 5 hours after the object is removed from the furnace.

Glossary

Note: The number in parentheses indicates the lesson in which the concept or term is introduced.

absolute maximum values (12): *See* **extreme values of a function**.

absolute minimum values (12): *See* **extreme values of a function**.

absolute value function (3): The absolute value function is defined by $|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$.

accumulation function (21): A function of the form $G(x) = \int_a^x f(t)\,dt$.

antiderivative (18): A function G is an antiderivative of f on an interval I if $G'(x) = f(x)$ for all x in I.

area of a region in the plane (19): Let f be continuous and nonnegative on the interval $[a,b]$. Partition the interval into n equal subintervals of length $\Delta x = \dfrac{b-a}{n}$, $x_0 = a$, $x_1, x_2, \ldots, x_{n-1}, x_n = b$. The area of the region bounded by f, the x-axis, and the vertical lines $x = a$ and $x = b$ is $A = \lim\limits_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x$, $x_{i-1} \leq c_i \leq x_i$, provided this limit exists.

average value (20): *See* **mean value theorem for integrals** in the Theorems section.

axis of revolution (30): If a region in the plane is revolved about a line, the resulting solid is a solid of revolution, and the line is called the axis of revolution.

catenary (32): When a uniform, flexible cable, such as a telephone wire, is suspended from 2 points, it takes the shape of a catenary. (The St. Louis Arch is also in the shape of a catenary.) The formula for a typical catenary is $y = 75\left(e^{\frac{x}{150}} + e^{-\frac{x}{150}}\right)$, $-100 \leq x \leq 100$.

compound interest formula (27): Let P be the amount of a deposit at an annual interest rate of r (as a decimal) compounded n times per year. The amount after t years is $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$. If the interest is compounded continuously, the amount is $A = Pe^{rt}$.

concavity (14): Let f be differentiable on an open interval I. The graph of f is concave upward on I if f' is increasing on I and concave downward on I if f' is decreasing on I. A graph is concave upward if the graph is above its tangent lines and concave downward if the graph is below its tangent lines.

continuous function (5): A function f is continuous at c if the following 3 conditions are met: $f(c)$ is defined, $\lim\limits_{x \to c} f(x)$ exists, and $\lim\limits_{x \to c} f(x) = f(c)$.

critical number (12): Let f be defined at c. If $f'(c) = 0$ or f is not differentiable at c, then c is a critical number of f.

decreasing function (13): A function f is decreasing on an interval I if for any 2 numbers a and b in the interval, $a < b$ implies that $f(a) > f(b)$.

definite integral (19): Let f be defined on the interval $[a,b]$. Partition the interval into n equal subintervals of length $\Delta x = \dfrac{b-a}{n}$, $x_0 = a, x_1, x_2, \ldots, x_{n-1}$, $x_n = b$. Assume that the following limit exists: $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$, where $x_{i-1} \le c_i \le x_i$. Then this limit is the definite integral of f from a to b and is denoted $\int_a^b f(x)\,dx$.

Δx (2, 7): Δx (read "delta x") denotes a (small) change in x. Some textbooks use h instead of Δx.

derivative (7): The derivative of f at x is given by the following limit, if it exists:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Notations for the derivative of $y = f(x)$:

$$f'(x),\ \frac{dy}{dx},\ y',\ \frac{d}{dx}[f(x)],\ D[y].$$

The definitions of slope and the derivative are based on the difference quotient for slope (7):

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$

differential (15): Let $y = f(x)$ be a differentiable function. Then $dx = \Delta x$ is called the differential of x. The differential of y is $dy = f'(x)dx$.

differential equation (9, 35): A differential equation in x and y is an equation that involves x, y, and derivatives of y. The order of a differential equation is determined by the highest-order derivative in the equation.

differential of arc length (32): The differential of arc length is $ds = \sqrt{1 + f'(x)^2}\,dx$.

domain (2): *See* **function**.

***e* (24)**: The irrational number e is defined to be the unique number satisfying $\int_1^e \frac{1}{t}\,dt = 1$; that is, $\ln e = 1$. $e \approx 2.7182818$.

even function (3): A function f is even if $f(-x) = f(x)$.

exponential function (26): The inverse of the natural logarithmic function $y = \ln x$ is the exponential function $y = e^x$. The exponential function is equal to its derivative: $\frac{d}{dx}\left[e^x\right] = e^x$.

exponential function to base *a* (27): The exponential function to base a, $a > 0$, is defined by $a^x = e^{(\ln a)x}$.

extreme values (or extrema) of a function (12): Let f be defined on an open interval I containing c. We say $f(c)$ is the minimum of f on I if $f(c) \le f(x)$ for all x in I. Similarly, $f(c)$ is the maximum of f on I if $f(c) \ge f(x)$ for all x in I. These are the extreme values, or extrema, of f. We sometimes say these are the absolute minimum and maximum values of f on I.

function (2): Given 2 sets A and B, a function f is a correspondence that assigns to each number x in A exactly 1 number y in B. The set A is the domain of the function. The number y is the image of x under f and is denoted by $f(x)$. The range of f is the subset of B consisting of all the images.

greatest integer function (5): The greatest integer function is defined as follows: $[\![x]\!] = \text{the greatest integer } n \text{ such that } n \le x.$

growth and decay model (36): The solution to the growth and decay model $\frac{dy}{dt} = ky$ is $y = Ce^{kt}$.

horizontal asymptote (6): The line $y = L$ is a horizontal asymptote of the graph of f if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L.$

improper integral (29): An integral of the form $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$.

increasing function (13): A function f is increasing on an interval I if for any 2 numbers a and b in the interval, $a < b$ implies that $f(a) < f(b)$.

infinite limit (6): The equation $\lim_{x \to c} f(x) = \infty$ means that $f(x)$ increases without bound as x approaches c.

integration by substitution (22): Let F be an antiderivative of f. If $u = g(x)$, then $du = g'(x)dx$, so we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \text{ because}$$
$$\int f(u)\,du = F(u) + C.$$

intercepts (2): The intercepts of a graph are the points where the graph intersects the x- or y-axis.

inverse function (25): A function g is the inverse function of the function f if $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for all x in the domain of f. The inverse of f is denoted f^{-1}.

inverse trigonometric functions (28): These are defined as follows:

$$y = \arcsin x = \sin^{-1} x \Leftrightarrow \sin y = x, \text{ for } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

$$y = \arccos x = \cos^{-1} x \Leftrightarrow \cos y = x, \text{ for } -1 \le x \le 1 \text{ and } 0 \le y \le \pi.$$

$$y = \arctan x = \tan^{-1} x \Leftrightarrow \tan y = x, \text{ for } -\infty < x < \infty \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

$$y = \text{arcsec}\, x = \sec^{-1} x \Leftrightarrow \sec y = x, \text{ for } |x| \ge 1,\ 0 \le y \le \pi, \text{ and } y \ne \frac{\pi}{2}.$$

limit, formal definition (4): Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

limit, informal definition (4): If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, we say that the limit of $f(x)$ as x approaches c is L, which we write as $\lim_{x \to c} f(x) = L$.

local maximum (12): *See* **relative maximum**.

local minimum (12): *See* **relative minimum**.

log rule for integration (24): $\int \frac{1}{x}\,dx = \ln|x| + C$.

logarithmic function to base *a* (27): For $a > 0$ and $a \neq 1$, $\log_a x = \dfrac{1}{\ln a} \ln x$.

lower sum (19): The sum $s(n)$ of the areas of n inscribed rectangles.

maximum values (12): *See* **extreme values of a function**.

minimum values (12): *See* **extreme values of a function**.

natural logarithmic function (24): The natural logarithmic function is defined by the definite integral $\ln x = \int_1^x \frac{1}{t} dt,\ x > 0$.

net change (21): *See* **Definitions and Theorems** in Lesson Twenty-One.

Newton's law of cooling (36): The rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

odd function (3): A function f is odd if $f(-x) = -f(x)$.

one-sided limits (5): The limit from the right means that x approaches c from values greater than c. The notation is $\lim_{x \to c^+} f(x) = L$. Similarly, the limit from the left means that x approaches c from values less than c, notated $\lim_{x \to c^-} f(x) = L$.

one-to-one function (3): A function from X to Y is one-to-one if to each y-value in the range, there corresponds exactly 1 x-value in the domain.

point of inflection (14): A point $(c, f(c))$ is a point of inflection (or inflection point) if the concavity changes at that point.

point of intersection (2): A point of intersection of the graphs of 2 equations is a point that satisfies both equations.

point-slope equation (1): The point-slope equation of the line passing through the point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.

radians (3): Calculus uses radian measure. If a problem is stated in degree measure, you must convert to radians: 360° is 2π radians; 180° is π radians.

range (2): *See* **function**.

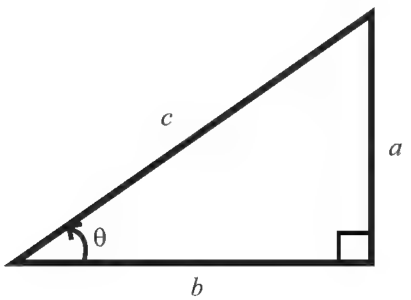
relative maximum (12): If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum (or local maximum) of f.

relative minimum (12): If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a relative minimum (or local minimum) of f.

representative rectangle (29): If $f(x) \geq g(x)$ on the interval $[a, b]$, we say that the small rectangle of width Δx and height $f(x) - g(x)$ is a representative rectangle.

Riemann sum (19): The expression $\sum_{i=1}^{n} f(c_i)\Delta x$.

right-triangle definition of the trigonometric functions (3): The right-triangle definition of the trigonometric functions uses the right triangle below.



$$\sin\theta = \frac{a}{c}$$
$$\cos\theta = \frac{b}{c}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a}{b}$$

sigma notation for sums (19): $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

slope (1): The slope m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, where $x_1 \neq x_2$.

slope field (35): Given a differential equation in x and y, $y' = F(x, y)$, the equation determines the derivative at each point (x, y). If you draw short line segments with slope $F(x, y)$ at selected points (x, y), then these line segments form a slope field.

slope of the graph of a function (7): The slope of the graph of the function f at the point $(c, f(c))$ is defined as the following limit, if it exists: $m = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$. By letting $x = c + \Delta x$, so $x - c = \Delta x$, you obtain the equivalent definition of slope: $m = \lim_{\Delta x \to 0} \dfrac{f(c + \Delta x) - f(c)}{\Delta x}$.

smooth (32): The graph of a function f is smooth on the interval $[a, b]$ if f' is continuous on that interval.

solid of revolution (30): If a region in the plane is revolved about a line, the resulting solid is a solid of revolution, and the line is called the axis of revolution.

solution curves (35): The general solution of a first-order differential equation represents a family of curves known as solution curves, one for each value of the arbitrary constant.

symmetric with respect to the origin (2): A graph in which if (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph.

symmetric with respect to the *x*-axis (2): A graph in which if (x, y) is a point on the graph, $(x, -y)$ is also a point on the graph.

symmetric with respect to the *y*-axis (2): A graph in which if (x, y) is a point on the graph, $(-x, y)$ is also a point on the graph.

tangent line problem (1): The calculation of the slope of the tangent line to a curve at a specific point.

torus (31): A surface or solid shaped like a tire or doughnut and formed by revolving the region bounded by the circle $x^2 + y^2 = r^2$ about the line $x = R$ $(r < R)$.

total change (21): *See* **Definitions and Theorems** in Lesson Twenty-One.

total distance traveled (21): *See* **Definitions and Theorems** in Lesson Twenty-One.

unit-circle definition (3): Let (x, y) be a point on the unit circle $x^2 + y^2 = 1$. The unit-circle definition of the trigonometric functions is

$$\sin\theta = y,\ \cos\theta = x,\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}.$$

$$\csc\theta = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}.$$

upper sum (19): The sum $S(n)$ of the areas of n circumscribed rectangles.

vertical asymptote (6): If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or left, then the vertical line $x = c$ is a vertical asymptote of the graph of f.

vertical line test (2): A vertical line can intersect the graph of a function at most once. If a vertical line intersects the graph at more than 1 point, the graph is not a function.

Formulas

Summary of Differentiation Formulas

1. Constant multiple rule: $\frac{d}{dx}[cu] = cu'$.

2. Sum or difference rule: $\frac{d}{dx}[u \pm v] = u' \pm v'$.

3. Product rule: $\frac{d}{dx}[uv] = uv' + vu'$.

4. Quotient rule: $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$.

5. Constant rule: $\frac{d}{dx}[c] = 0$.

6. Chain rule: $\frac{d}{dx}[f(u)] = f'(u)u'$.

7. General power rule: $\frac{d}{dx}[u^n] = nu^{n-1}u'$.

8. $\frac{d}{dx}[x] = 1$.

9. $\frac{d}{dx}[\ln x] = \frac{1}{x}$.

10. $\frac{d}{dx}[e^x] = e^x$.

11. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$.

12. $\frac{d}{dx}[a^x] = (\ln a)a^x$.

13. $\frac{d}{dx}[\sin x] = \cos x$.

14. $\frac{d}{dx}[\cos x] = -\sin x$.

15. $\frac{d}{dx}[\tan x] = \sec^2 x$.

16. $\frac{d}{dx}[\cot x] = -\csc^2 x$.

17. $\frac{d}{dx}[\sec x] = \sec x \tan x$.

18. $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$.

19. $\dfrac{d}{dx}[\arcsin x] = \dfrac{1}{\sqrt{1-x^2}}$.

20. $\dfrac{d}{dx}[\arctan x] = \dfrac{1}{1+x^2}$.

21. $\dfrac{d}{dx}[\operatorname{arc\,sec} x] = \dfrac{1}{|x|\sqrt{x^2-1}}$.

Summary of Integration Formulas

1. $\int kf(x)dx \;=\; k\int f(x)dx$.

2. $\int [f(x) \pm g(x)]dx \;=\; \int f(x)dx \;\pm\; \int g(x)dx$.

3. $\int d(x) \;=\; x + C$.

4. Power rule for integration: $\int x^n dx \;=\; \dfrac{x^{n+1}}{n+1} + C$, for $n \neq -1$.

5. Log rule for integration: $\int \dfrac{dx}{x} \;=\; \ln|x| + C$.

6. $\int e^x dx \;=\; e^x + C$.

7. $\int a^x dx \;=\; \left(\dfrac{1}{\ln a}\right)a^x + C$.

8. $\int \sin x dx \;=\; -\cos x + C$.

9. $\int \cos x dx \;=\; \sin x + C$.

10. $\int \tan x dx \;=\; -\ln|\cos x| + C$.

11. $\int \cot x dx \;=\; \ln|\sin x| + C$.

12. $\int \sec x dx \;=\; \ln|\sec x + \tan x| + C$.

13. $\int \csc x dx \;=\; -\ln|\csc x + \cot x| + C$.

14. $\int \sec^2 x dx \;=\; \tan x + C$.

15. $\int \csc^2 x dx \;=\; -\cot x + C$.

16. $\int \sec x \tan x dx \;=\; \sec x + C$.

17. $\int \csc x \cot x dx = -\csc x + C$.

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$.

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$.

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|x|}{a} + C$.

Other Formulas

arc length formula (32): Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a,b]$. The arc length of f between a and b is $s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

compound interest formulas (27): Let P be the amount of a deposit at an annual interest rate of r (as a decimal) compounded n times per year. The amount after t years is $A = P\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, the amount is $A = Pe^{rt}$.

summation formulas (19):

$$\sum_{i=1}^{n} c = c + c + \cdots + c = cn.$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Theorems

Note: The number in parentheses indicates the lesson in which the theorem is introduced.

differentiability implies continuity (7): If f is differentiable at $x = c$, then f is continuous at $x = c$.

extreme value theorem (12): If f is continuous on a closed interval $[a,b]$, then f has both a minimum and a maximum on the interval.

first derivative test (13): Let c be a critical number of f. If f' changes from positive to negative at c, then f has a relative maximum at $(c, f(c))$. If f' changes from negative to positive at c, then f has a relative minimum at $(c, f(c))$.

fundamental theorem of calculus (20): Let f be continuous on the closed interval $[a,b]$. Let G be an antiderivative of f. Then $\int_a^b f(x)\,dx = G(b) - G(a)$.

integration by parts (34): $\int u\,dv = uv - \int v\,du$.

intermediate value theorem (5): If f is continuous on the closed interval $[a,b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least 1 number c in $[a,b]$ such that $f(c) = k$.

mean value theorem for derivatives (13): Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Then there is at least 1 number c in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

mean value theorem for integrals (20): Let f be continuous on the closed interval $[a,b]$. The mean value theorem for integrals says that there exists a number c in $[a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$. The value $f(c)$ is called the average value of f.

relative extrema at critical numbers theorem (12): If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

Rolle's theorem (13): Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least 1 number c in (a,b) such that $f'(c) = 0$.

second derivative test (14): Let $f'(c) = 0$ (c is a critical number of f). If $f''(c) > 0$, then f has a relative minimum at c. If $f''(c) < 0$, then f has a relative maximum at c.

second fundamental theorem of calculus (21): Let f be continuous on an open interval I containing a. The second fundamental theorem of calculus says that for any x in the interval, $\dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

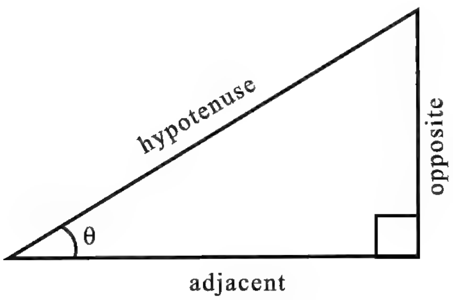
trapezoidal rule (23): Let f be continuous on the interval $[a,b]$. Partition the interval into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$: $a = x_0 < x_1 < x_2 < \cdots x_{n-1} < x_n = b$. The trapezoidal rule for approximating the definite integral of f between a and b is given by

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right].$$

Review of Trigonometry

Definition of the 6 Trigonometric Functions

Right triangle definitions, where $0 < \theta < \frac{\pi}{2}$.

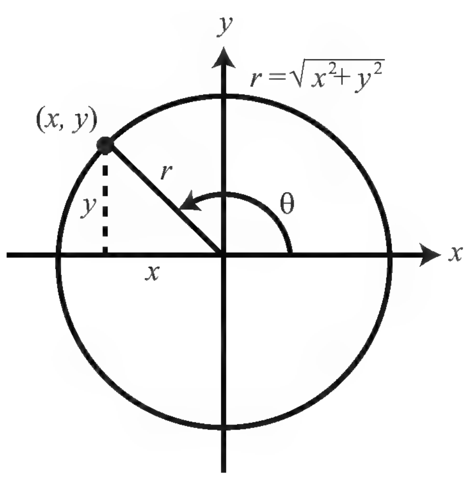


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Circular function definitions, where θ is any angle.



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Reciprocal Identities

$$\sin x = \frac{1}{\csc x}. \qquad \sec x = \frac{1}{\cos x}. \qquad \tan x = \frac{1}{\cot x}.$$

$$\csc x = \frac{1}{\sin x}. \qquad \cos x = \frac{1}{\sec x}. \qquad \cot x = \frac{1}{\tan x}.$$

Tangent and Cotangent Identities

$$\tan x = \frac{\sin x}{\cos x}. \qquad \cot x = \frac{\cos x}{\sin x}.$$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1.$
$1 + \tan^2 x = \sec^2 x.$
$1 + \cot^2 x = \csc^2 x.$

Reduction Formulas

$\sin(-x) = -\sin x.$ $\cos(-x) = \cos x.$
$\csc(-x) = -\csc x.$ $\tan(-x) = -\tan x.$
$\sec(-x) = \sec x.$ $\cot(-x) = -\cot x.$

Sum and Difference Formulas

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v.$$
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v.$$
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}.$$

Double-Angle Formulas

$\sin 2u = 2\sin u \cos u.$
$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u.$

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}.$$
$$\cos^2 u = \frac{1 + \cos 2u}{2}.$$

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Solutions

Lesson One

1. Find the equation of the tangent line to the parabola $y = x^2$ at the point $(3,9)$.

Let $Q = (x, x^2)$ be another point on the graph of the parabola $y = x^2$. The slope of the line joining $(3,9)$ and (x, x^2) is $m = \dfrac{x^2 - 9}{x - 3} = \dfrac{(x-3)(x+3)}{x-3} = x + 3, x \neq 3$. As Q approaches $(3,9)$, x approaches 3, and the slope approaches 6. Hence the slope of the tangent line is 6, and the equation becomes $y - 9 = 6(x - 3)$, or $y = 6x - 9$.

2. Find the equation of the tangent line to the parabola $y = x^2$ at the point $(0,0)$.

Let $Q = (x, x^2)$ be another point on the graph of the parabola $y = x^2$. The slope of the line joining $(0,0)$ and (x, x^2) is $m = \dfrac{x^2 - 0}{x - 0} = \dfrac{x^2}{x} = x, x \neq 0$. As Q approaches $(0,0)$, x approaches 0, and the slope approaches 0. Hence the slope of the tangent line is 0, and the equation becomes $y - 0 = 0(x - 0)$, or $y = 0$. The tangent line is horizontal!

3. Find the equation of the tangent line to the cubic polynomial $y = x^3$ at the point $(-1,-1)$.

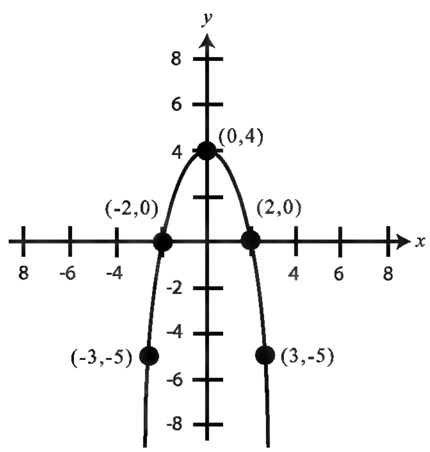
Let $Q = (x, x^3)$ be another point on the graph of the cubic polynomial $y = x^3$. The slope of the line joining $(-1,-1)$ and (x, x^3) is $m = \dfrac{x^3 - (-1)}{x - (-1)} = \dfrac{x^3 + 1}{x + 1} = \dfrac{(x+1)(x^2 - x + 1)}{x + 1} = x^2 - x + 1, x \neq -1$.

Notice how we factored the numerator, which is a sum of 2 cubics. As Q approaches $(-1,-1)$, x approaches -1, and the slope approaches $(-1)^2 - (-1) + 1 = 3$. Hence, the slope of the tangent line is 3, and the equation becomes $y - (-1) = 3(x - (-1))$, or $y = 3x + 2$.

Lesson Two

1. Sketch the graph of the equation $y = 4 - x^2$ by point plotting.

x	-3	-2	0	2	3
y	-5	0	4	0	-5



2. Find the intercepts of the graph of the equation $x\sqrt{16 - x^2}$.

y-intercept: $y = 0\sqrt{16 - 0^2} = 0; (0,0)$.

x-intercepts: $0 = x\sqrt{16 - x^2} = x\sqrt{(4-x)(4+x)}$.
$x = 0, 4, -4; (0,0), (4,0), (-4,0)$.

3. Test the equation $y = \dfrac{x^2}{x^2 + 1}$ for symmetry with respect to each axis and the origin.

It is symmetric with respect to the y-axis, because $y = \dfrac{(-x)^2}{(-x)^2 + 1} = \dfrac{x^2}{x^2 + 1}$.

4. Find the points of intersection of the graphs of $x^2 + y = 6$ and $x + y = 4$. Verify your answer with a graphing utility.
Solve for y in both equations: $y = 6 - x^2$; $y = 4 - x$. Setting these equal to each other, $6 - x^2 = 4 - x \Rightarrow x^2 - x - 2 = 0$. Factoring, $(x-2)(x+1) = 0 \Rightarrow x = 2, -1$. The corresponding y-values are $y = 2$ (for $x = 2$) and $y = 5$ (for $x = -1$). The points of intersection are $(2,2)$ and $(-1,5)$.

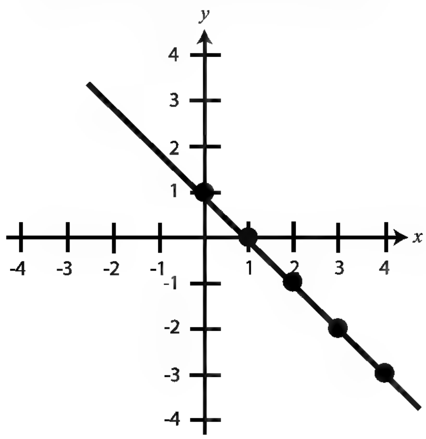
5. Find the slope of the line passing through the points $(3,-4)$ and $(5,2)$.

$$m = \frac{2-(-4)}{5-3} = \frac{6}{2} = 3.$$

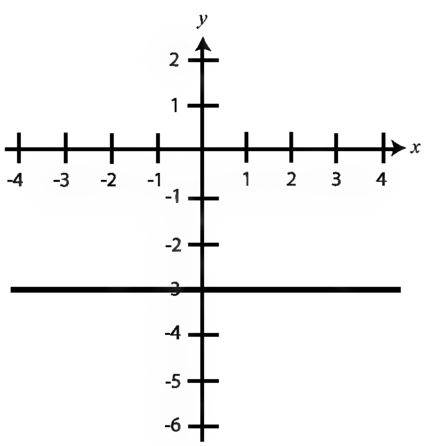
6. Determine an equation of the line that passes through the points $(2,1)$ and $(0,-3)$.

$$m = \frac{1-(-3)}{2-0} = 2;\ y - 1 = 2(x-2) \Rightarrow y = 2x - 3.$$

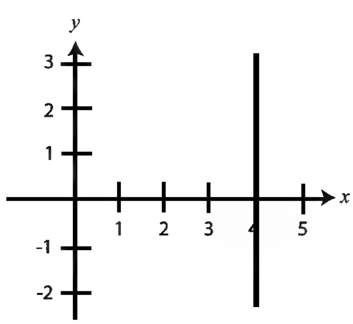
7. Sketch the graphs of the equations $x+y=1$, $y=-3$, and $x=4$.



$x+y=1$



$y=-3$



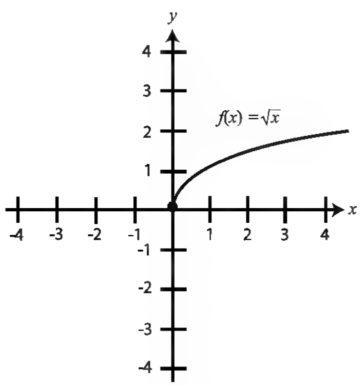
$x=4$

8. Find an equation of the line that passes through the point $(2,1)$ and is perpendicular to the line $5x-3y=0$.

The equation of the given line is $y=\frac{5}{3}x$, so the slope of the perpendicular line is $\frac{-3}{5}$. The equation is $y-1=\frac{-3}{5}(x-2)$, or $3x+5y=11$.

9. Determine the domain and range of the square root function $f(x)=\sqrt{x}$. Sketch its graph.

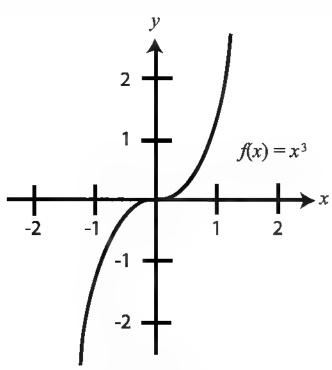
Because you cannot take the square root of a negative number, the domain is $x \ge 0$. The range is $y \ge 0$.



Square root function

10. Determine the domain and range of the function $f(x)=x^3$. Sketch its graph.

Both the domain and range are all real numbers.



Cubing function

Lesson Three

1. Find the domain and range of the function $f(x) = x^2 - 5$.

The domain is all real numbers. Since f is a vertical shift of -5, the range is $y \geq -5$.

2. Find the domain and range of the function $f(x) = -\sqrt{x+3}$.

The domain is $x \geq -3$. The range is $y \leq 0$.

3. Find the domain and range of the function $f(x) = \cot x$.

$\cot x = \dfrac{\cos x}{\sin x}$ is defined for all real numbers except where sine equals 0. Hence the domain is all real numbers $x \neq n\pi$, where n is an integer. The range is all real numbers.

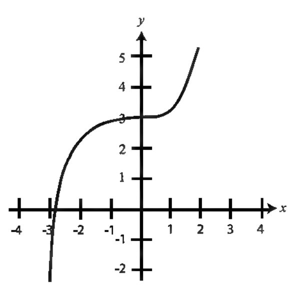
4. Evaluate the function

$$f(x) = \begin{cases} |x| + 1, x < 1 \\ -x + 1, x \geq 1 \end{cases}$$

at the points $x = -3,\ 1, 3,$ and $b^2 + 1$.

$f(-3) = |-3| + 1 = 4; f(1) = -1 + 1 = 0; f(3) = -3 + 1 = -2;$
and since $b^2 + 1 \geq 1$, $f(b^2 + 1) = -(b^2 + 1) + 1 = -b^2$.

5. Sketch the graph of the function $f(x) = \dfrac{1}{3}x^3 + 3$ and find its domain and range.



Both the domain and range consist of all real numbers.

6. Determine whether the function $f(x) = \sqrt[3]{x}$ is even or odd.

f is odd because $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$.

7. Determine whether the function $f(x) = x\cos x$ is even or odd.

f is odd because $f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$.

8. Find the values of the 6 trigonometric functions corresponding to the angle $\dfrac{\pi}{6}$.

Using a unit circle, mark off the angle $\dfrac{\pi}{6}$. Then $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$, and $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$.

Thus, $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$, $\cot\dfrac{\pi}{6} = \sqrt{3}$, $\sec\dfrac{\pi}{6} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$, and $\csc\dfrac{\pi}{6} = 2$.

9. Use the identity $\sin^2 x + \cos^2 x = 1$ to derive the identity $\tan^2 x + 1 = \sec^2 x$.

Divide both sides of the fundamental identity by $\cos^2 x$:

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x.$$

10. Find all solutions to the trigonometric equation $2\sin^2 x - \sin x - 1 = 0$ on the interval $[0, 2\pi)$.

Factor the expression $(2\sin x + 1)(\sin x - 1) = 0 \Rightarrow \sin x = \frac{-1}{2}$, or $\sin x = 1$. The solutions to the first equation are $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$. The solution to the second equation is $x = \frac{\pi}{2}$.

11. Find all solutions to the trigonometric equation $\tan x = 0$.

$\tan x = \frac{\sin x}{\cos x} = 0$ when $\sin x = 0$. Hence the solutions are $x = n\pi$, where n is an integer.

Lesson Four

1. Find the limit: $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$. Verify your answer with a graphing utility.

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{x^2-3^2}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6.$$

2. Find the limit: $\lim\limits_{x\to\pi}\sin(2x)$.

$$\lim_{x\to\pi}\sin(2x)=\sin(2\pi)=0.$$

3. Find the limit: $\lim\limits_{x\to 4}\sqrt[3]{x+4}$.

$$\lim_{x\to 4}\sqrt[3]{x+4}=\sqrt[3]{4+4}=\sqrt[3]{8}=2.$$

4. Find the limit: $\lim\limits_{x\to -3}\dfrac{x^2+x-6}{x+3}$. Verify your answer with a table of values.

$$\lim_{x\to -3}\frac{x^2+x-6}{x+3}=\lim_{x\to -3}\frac{(x+3)(x-2)}{x+3}=\lim_{x\to -3}(x-2)=-5.$$

5. Find the limit: $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$.

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\frac{1}{\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x}=(1)(1)=1.$$

6. Find the limit: $\lim\limits_{x\to 4}\dfrac{\sqrt{x+5}-3}{x-4}$.

$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}&=\lim_{x\to 4}\frac{(\sqrt{x+5}-3)}{(x-4)}\frac{(\sqrt{x+5}+3)}{(\sqrt{x+5}+3)}\\&=\lim_{x\to 4}\frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)}\\&=\lim_{x\to 4}\frac{x-4}{(x-4)\left(\sqrt{x+5}+3\right)}\\&=\lim_{x\to 4}\frac{1}{\left(\sqrt{x+5}+3\right)}=\frac{1}{6}.\end{aligned}$$

7. Discuss the existence of the following limit: $\lim\limits_{x\to -4}\dfrac{x+4}{|x+4|}$.

As x approaches -4 from the left, the function equals -1, whereas from the right it equals 1. The function is not approaching a single number L as x approaches -4. The limit does not exist.

8. Discuss the existence of the following limit: $\lim\limits_{x\to 2}\dfrac{3}{x-2}$.

As x approaches 2 from the right, the values of the function increase without bound. Similarly, as x approaches 2 from the left, the values decrease without bound. The function is not approaching a single number L as x approaches 2. The limit does not exist.

9. Find the following limit: $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}$.

Notice in the following that $x \to 0$ is equivalent to $5x = u \to 0$.

$$\lim_{x \to 0} \frac{\sin 5x}{x} = 5 \lim_{x \to 0} \frac{\sin 5x}{5x} = 5 \lim_{u \to 0} \frac{\sin u}{u} = 5(1) = 5.$$

10. Discuss the existence of $\lim_{x \to 3} f(x)$, where

$$f(x) = \begin{cases} 5, x \neq 3 \\ 1, x = 3 \end{cases}.$$

The value of the function at $x = 3$ is irrelevant. As x approaches 3, the function always equals 5. The limit is 5.

Lesson Five

1. Find the following limit: $\lim_{x \to 0^-} \frac{|x|}{x}$.

   As x approaches 0 from the right, the value of the function is 1. Hence the 1-sided limit is 1.

2. Find the x-values, if any, at which the function $f(x) = \frac{6}{x}$ is not continuous. Which of the discontinuities are removable?

   The function is not continuous at 0 because 0 is not in the domain. The discontinuity is not removable. There is no value to assign to $f(0)$ that will make the function continuous there.

3. Find the x-values, if any, at which the function $f(x) = x^2 - 9$ is not continuous. Which of the discontinuities are removable?

   This function is a polynomial and therefore continuous for all values of x.

4. Find the x-values, if any, at which the function $f(x) = \frac{x-6}{x^2-36}$ is not continuous. Which of the discontinuities are removable?

   The function is not continuous at $x = \pm 6$. By factoring the denominator, you see that $\frac{x-6}{x^2-36} = \frac{x-6}{(x-6)(x+6)} = \frac{1}{x+6}, x \neq 6$. The discontinuity at -6 is nonremovable, but the discontinuity at 6 is removable by defining that $f(6) = \frac{1}{6+6} = \frac{1}{12}$.

5. Discuss the continuity of the function $f(x) = \frac{x^2-1}{x+1}$.

   The function is not continuous at -1. This discontinuity is removable because $\frac{x^2-1}{x+1} = \frac{(x-1)(x+1)}{x+1} = x-1, x \neq -1,$ indicates that we should define that $f(-1) = -1-1 = -2$.

6. Discuss the continuity of the function $f(x) = \sqrt{49 - x^2}$.

   The function is clearly continuous on the open interval $(-7,7)$. Furthermore, the left- and right-hand limits exist at the endpoints 7 and -7. Thus, the function is continuous on its entire domain, $[-7,7]$.

7. Explain why the function $f(x) = x^3 + 5x - 3$ has a zero in the interval $[0,1]$.

   The function is continuous everywhere: $f(0) = -3$, and $f(1) = 3$. By the intermediate value theorem, there exists a number c in the interval $[0,1]$ satisfying $f(c) = 0$.

8. Explain why the function $f(x) = x^2 - 2 - \cos x$ has a zero in the interval $[0,\pi]$.

   The function is continuous everywhere: $f(0) = -3$, and $f(\pi) = \pi^2 - 2 + 1 > 0$. By the intermediate value theorem, there exists a number c in the interval $[0,\pi]$ satisfying $f(c) = 0$.

9. True or false: The function $f(x) = \frac{|x-1|}{x-1}$ is continuous on $(-\infty,\infty)$.

   False. The function is not defined at 1, so it is not continuous there. In fact, the discontinuity is nonremovable.

10. Discuss the continuity of the function

$$f(x)=\begin{cases}-2x+3, & x<1\\ x^2, & x\geq 1\end{cases}.$$

This function is continuous everywhere, except possibly at $x=1$. But the limit from the left at $x=1$ equals 1, which is the same as the limit from the right at $x=1$. Therefore, the function is continuous for all values of x.

Lesson Six

1. Determine whether $f(x) = \dfrac{1}{x-4}$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

As x approaches 4 from the left, $x-4$ is negative and approaches 0, and hence $\dfrac{1}{x-4}$ approaches $-\infty$.

Similarly, as x approaches 4 from the right, $x-4$ is positive and approaches 0, so $\dfrac{1}{x-4}$ approaches ∞.

2. Determine whether $f(x) = \dfrac{-1}{(x-4)^2}$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.

As x approaches 4 from the left or the right, $(x-4)^2$ is positive and approaches 0. Hence in both cases, $\dfrac{-1}{(x-4)^2}$ approaches $-\infty$.

3. Find the vertical asymptotes, if any, of the function $f(x) = \dfrac{x^2}{x^2-4}$.

The zeros of the denominator are ± 2, and these are not zeros of the numerator. Hence, there are vertical asymptotes at $x = \pm 2$.

4. Find the vertical asymptotes, if any, of the function $f(x) = \dfrac{x^2-1}{x+1}$.

The zero of the denominator, $x = -1$, is also a zero of the numerator. Therefore, there are no vertical asymptotes. In fact, there is a hole in the graph of f at $(-1, -2)$.

5. Find the limit: $\lim\limits_{x \to -3^-} \dfrac{x+3}{x^2+x-6}$.

$$\lim_{x \to -3^-} \frac{x+3}{x^2+x-6} = \lim_{x \to -3^-} \frac{x+3}{(x+3)(x-2)} = \lim_{x \to -3^-} \left(\frac{1}{x-2}\right) = \frac{1}{-3-2} = \frac{-1}{5}.$$

6. Find the limit: $\lim\limits_{x \to \infty} \dfrac{x^2+2}{x^3-1}$.

$\lim\limits_{x \to \infty} \dfrac{x^2+2}{x^3-1} = \lim\limits_{x \to \infty} \dfrac{x^2+2}{x^3-1}\left(\dfrac{\frac{1}{x^3}}{\frac{1}{x^3}}\right) = \lim\limits_{x \to \infty} \dfrac{\frac{1}{x}+\frac{2}{x^3}}{1-\frac{1}{x^3}} = \dfrac{0+0}{1-0} = \dfrac{0}{1} = 0$. Note that the degree of the denominator is greater than the degree of the numerator, so the limit is 0.

7. Find the limit: $\lim\limits_{x \to \infty} \dfrac{x^2+2}{x^2-1}$.

The degrees of the numerator and denominator are the same, so the limit is the ratio of the leading coefficients: $\lim\limits_{x \to \infty} \dfrac{x^2+2}{x^2-1} = \dfrac{1}{1} = 1$

8. Find the limit: $\lim_{x\to-\infty}\left(\frac{5}{x}-\frac{x}{3}\right)$.

The common denominator is $3x$: $\lim_{x\to-\infty}\left(\frac{5}{x}-\frac{x}{3}\right)=\lim_{x\to-\infty}\left(\frac{15-x^2}{3x}\right)=\infty$. Note that the degree of the numerator is greater than the degree of the denominator and that both are negative as $x\to-\infty$.

9. Find the limit: $\lim_{x\to\infty}\frac{\sin x}{x}$.

As x tends to infinity, the denominator increases without bound, whereas the numerator is bounded: $|\sin x|\le 1$. Hence the quotient tends to 0: $\lim_{x\to\infty}\frac{\sin x}{x}=0$.

10. Use a graphing utility to identify the horizontal asymptotes of the function $f(x)=\frac{3x-2}{\sqrt{2x^2+1}}$.

The graph shows 2 horizontal asymptotes, $\frac{3x-2}{\sqrt{2x^2+1}}\to\frac{-3}{\sqrt{2}}$ to the left and $\frac{3x-2}{\sqrt{2x^2+1}}\to\frac{3}{\sqrt{2}}$ to the right.

Lesson Seven

1. Find the slope of the tangent line to the graph of $f(x)=3-5x$ at the point $(-1,8)$.

Using the limit definition of slope, you have

$$m=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to -1}\frac{(3-5x)-8}{x-(-1)}=\lim_{x\to -1}\frac{-5(x+1)}{x+1}=\lim_{x\to -1}(-5)=-5.$$

This is the answer we expect, since the function f is a line with slope -5.

2. Find the slope of the tangent line to the graph of $f(x)=x^2$ at the point $(3,9)$.

Using the limit definition of slope, you have

$$m=\lim_{x\to 3}\frac{f(x)-f(c)}{x-c}=\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6.$$

3. Find the derivative of $f(x)=7$ by the limit process.

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{7-7}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0$$

This answer makes sense because the graph of f is a horizontal line.

4. Find the derivative of $f(x)=3x+2$ by the limit process.

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{[3(x+\Delta x)+2]-[3x+2]}{\Delta x}=\lim_{\Delta x\to 0}\frac{3\Delta x}{\Delta x}=\lim_{\Delta x\to 0}\frac{3}{1}=3$$

This answer makes sense because the graph of f is a line with slope 3.

5. Find the derivative of $f(x)=\dfrac{1}{x^2}$ by the limit process.

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\dfrac{1}{(x+\Delta x)^2}-\dfrac{1}{x^2}}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2-(x+\Delta x)^2}{\Delta x(x+\Delta x)^2x^2}$$
$$=\lim_{\Delta x\to 0}\frac{x^2-(x^2+2x\Delta x+\Delta x^2)}{\Delta x(x+\Delta x)^2x^2}=\lim_{\Delta x\to 0}\frac{-2x\Delta x-\Delta x^2}{\Delta x(x+\Delta x)^2x^2}=\lim_{\Delta x\to 0}\frac{-2x-\Delta x}{(x+\Delta x)^2x^2}=\frac{-2x}{x^4}=\frac{-2}{x^3}$$

6. Find the equation of the tangent line to the graph $f(x)=\sqrt{x}$ at the point $(1,1)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

One way to solve this problem is to use the formula for the slope of the tangent line. Also note how we factor the denominator to obtain the cancellation.

$$f'(x)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt{x}-1}{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}=\lim_{x\to 1}\frac{1}{\left(\sqrt{x}+1\right)}=\frac{1}{2}$$

Hence the slope is $\frac{1}{2}$, and the tangent line is $y-1=\frac{1}{2}(x-1)$, or $y=\frac{1}{2}x+\frac{1}{2}$. Your graph should show the curve and the tangent line intersecting at the point $(1,1)$.

7. Determine whether or not the function $f(x)=|x+7|$ is differentiable everywhere.

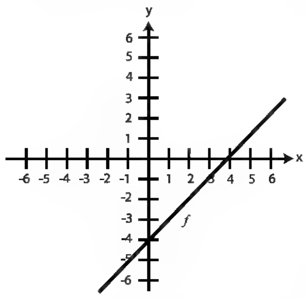
The function is not differentiable at the point $(-7,0)$. There is a sharp corner at that point. Using the definition of derivative, you see that the limit from the left is -1, whereas the limit from the right is 1.

8. Determine whether the function $f(x) = (x-6)^{\frac{2}{3}}$ is differentiable everywhere.

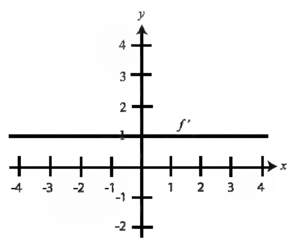
This function is not differentiable at 6 because the following limit does not exist:

$\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \lim\limits_{x \to 6} \dfrac{(x-6)^{2/3} - 0}{x - 6} = \lim\limits_{x \to 6} \dfrac{1}{(x-6)^{1/3}}$. If you graph the function, you will see a sharp corner, or cusp, at the point $(6,0)$.

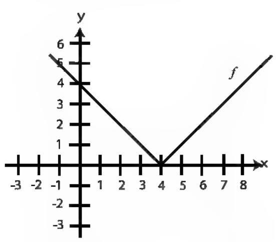
9. Given the graph of f, sketch the graph of f'.



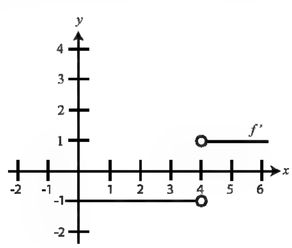
The slope of the given graph is approximately 1, so the graph of f' will be the horizontal line $y = 1$.



10. Given the graph of f, sketch the graph of f'.



The slope of the given graph is approximately -1 for $x < 4$ and 1 for $x > 4$. The derivative is not defined at the point $(4,0)$. Hence the graph of f' consists of 2 pieces, $y = -1\ (x < 4)$ and $y = 1\ (x > 4)$.



Lesson Eight

1. Find the derivative of $f(x)=\sqrt[5]{x}$.

$$\frac{d}{dx}\left[\sqrt[5]{x}\right]=\frac{d}{dx}\left[x^{\frac{1}{5}}\right]=\frac{1}{5}x^{\frac{1}{5}-1}=\frac{1}{5}x^{-\frac{4}{5}}=\frac{1}{5x^{4/5}}.$$

2. Find the derivative of $s(t)=t^3+5t^2-3t+8$.

$$\frac{d}{dt}\left[t^3+5t^2-3t+8\right]=3t^2+10t-3.$$

3. Find the derivative of $f(x)=\frac{1}{x^5}$.

$$\frac{d}{dx}\left[\frac{1}{x^5}\right]=\frac{d}{dx}\left[x^{-5}\right]=-5x^{-6}=\frac{-5}{x^6}.$$

4. Find the derivative of $f(x)=\frac{\sqrt{x}}{x}$.

$$\frac{d}{dx}\left[\frac{\sqrt{x}}{x}\right]=\frac{d}{dx}\left[x^{-\frac{1}{2}}\right]=\frac{-1}{2}x^{-\frac{3}{2}}=\frac{-1}{2x^{3/2}}.$$

5. Find the derivative of $f(x)=6\sqrt{x}+5\cos x$.

$$\frac{d}{dx}\left[6\sqrt{x}+5\cos x\right]=\frac{d}{dx}\left[6x^{\frac{1}{2}}+5\cos x\right]=6\left(\frac{1}{2}\right)x^{-\frac{1}{2}}-5\sin x=\frac{3}{\sqrt{x}}-5\sin x.$$

6. Find the equation of the tangent line to the graph $f(x)=3x^3-10$ at the point $(2,14)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

We see that $f'(x)=9x^2$, so the slope of the tangent line at the point $(2,14)$ is $9(2^2)=36$. The equation of the tangent line is $y-14=36(x-2)$, or $y=36x-58$. If you graph the original function and the tangent line in the same viewing window, you will see that the tangent line intersects the curve at the point $(2,14)$.

7. Find the equation of the tangent line to the graph $f(\theta)=4\sin\theta-\theta$ at the point $(0,0)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

We see that $f'(\theta)=4\cos\theta-1$, so the slope of the tangent line at the point $(0,0)$ is $4\cos 0-1=3$. The equation of the tangent line is $y-0=3(\theta-0)$, or $y=3\theta$. If you graph the original function and the tangent line in the same viewing window, you will see that the tangent line intersects the curve at the point $(0,0)$.

8. Find the equation of the tangent line to the graph $f(x)=(x^2+2)(x+1)$ at the point $(1,6)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

We must first write the function as a sum, not a product: $f(x)=(x^2+2)(x+1)=x^3+x^2+2x+2$. Then $f'(x)=3x^2+2x+2$, and $f'(1)=3(1^2)+2(1)+2=7$. The equation of the tangent line is $y-6=7(x-1)$, or $y=7x-1$. If you graph the original function and the tangent line in the same viewing window, you will see that the tangent line intersects the curve at the point $(1,6)$.

9. Determine the point(s) at which the graph of $f(x) = x^4 - 2x^2 + 3$ has a horizontal tangent line.

The graph will have a horizontal tangent at the points where the derivative is 0. $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1) = 0$. Hence, $x = 0,\ 1,$ and -1, and the points of horizontal tangency are $(0,3)$, $(-1,2)$, and $(1,2)$.

10. A silver dollar is dropped from a building that is 1362 feet tall. Determine the position and velocity functions for the coin. When does the coin hit the ground? Find the velocity of the coin at impact.

The acceleration due to gravity is –32, the initial velocity is 0, and the initial height is 1362. Hence, the position function of the coin is $s(t) = \frac{1}{2}gt^2 + v_0t + s_0 = -16t^2 + 1362$. The velocity is $v(t) = s'(t) = -32t$.

The coin hits the ground when $s(t) = -16t^2 + 1362 = 0$, which implies that $16t^2 = 1362$. Hence $t^2 = \frac{1362}{16}$, and $t = \frac{\sqrt{1362}}{4} \approx 9.226$ seconds. The velocity at impact is $v\left(\frac{\sqrt{1362}}{4}\right) = -32\left(\frac{\sqrt{1362}}{4}\right) =$

$-8\sqrt{1362} \approx 295.242$ feet per second.

Lesson Nine

1. Use the product rule to find the derivative of $f(x) = x^3 \cos x$.

$$f'(x) = x^3 \frac{d}{dx}[\cos x] + \cos x \frac{d}{dx}\left[x^3\right] = x^3(-\sin x) + \cos x\left(3x^2\right) = x^2(3\cos x - x\sin x).$$

2. Use the product rule to find the derivative of $f(x) = (x^2+3)(x^2-4x)$.

$$f'(x) = (x^2+3)(2x-4) + (x^2-4x)(2x) = (2x^3-4x^2+6x-12) + (2x^3-8x^2) = 4x^3-12x^2+6x-12.$$

Note that you could have expanded the function before differentiating and obtained the same answer.

3. Use the quotient rule to find the derivative of $f(x) = \dfrac{x}{x^2+1}$.

$$f'(x) = \frac{\left(x^2+1\right)\frac{d}{dx}[x] - x\frac{d}{dx}\left[x^2+1\right]}{\left(x^2+1\right)^2} = \frac{\left(x^2+1\right)(1) - x(2x)}{\left(x^2+1\right)^2} = \frac{1-x^2}{\left(x^2+1\right)^2}.$$

4. Use the quotient rule to find the derivative of $f(x) = \dfrac{\sin x}{x^2}$.

$$f'(x) = \frac{x^2(\cos x) - \sin x(2x)}{\left(x^2\right)^2} = \frac{x\cos x - 2\sin x}{x^3}.$$

5. Find the equation of the tangent line to the graph $f(x) = (x^3+4x-1)(x-2)$ at the point $(1,-4)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

$f'(x) = \left(x^3+4x-1\right)(1) + (x-2)(3x^2+4)$, and $f'(1) = 4-7 = -3$. The equation of the tangent line is $y+4 = -3(x-1)$, or $y = -3x-1$. If you graph the original function together with the tangent line, you will see that they intersect at the point $(1,-4)$.

6. Find the equation of the tangent line to the graph $f(x) = \dfrac{x-1}{x+1}$ at the point $\left(2, \dfrac{1}{3}\right)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

$f'(x) = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$, and $f'(2) = \dfrac{2}{9}$. The equation of the tangent line is $y - \dfrac{1}{3} = \dfrac{2}{9}(x-2)$, or $y = \dfrac{2}{9}x - \dfrac{1}{9}$. If you graph the original function together with the tangent line, you will see that they intersect at the point $\left(2, \dfrac{1}{3}\right)$.

7. Find the derivative of $f(x) = \dfrac{10}{3x^3}$ without the quotient rule.

$$\frac{d}{dx}\left[\frac{10}{3x^3}\right] = \frac{10}{3}\frac{d}{dx}\left[x^{-3}\right] = \frac{10}{3}(-3x^{-4}) = \frac{-10}{x^4}.$$

8. Find the derivative of the trigonometric function $f(x) = x^2 \tan x$.

$$f'(x) = x^2\sec^2 x + (\tan x)(2x) = x(2\tan x + x\sec^2 x).$$

9. Find the derivative of the trigonometric function $f(x)=\dfrac{\sec x}{x}$.

$$f'(x)=\frac{x\sec x\tan x-(\sec x)(1)}{x^2}=\frac{\sec x(x\tan x-1)}{x^2}.$$

10. Find the second derivative of $f(x)=4x^{\frac{3}{2}}$.

$$f'(x)=4\frac{3}{2}x^{\frac{1}{2}}=6x^{\frac{1}{2}},\text{ and } f''(x)=3x^{-\frac{1}{2}}=\frac{3}{\sqrt{x}}.$$

11. Find the second derivative of $f(x)=\dfrac{x}{x-1}$.

$$f'(x)=\frac{(x-1)(1)-x(1)}{(x-1)^2}=\frac{-1}{(x-1)^2}=-(x-1)^{-2},\text{ and } f''(x)=2(x-1)^{-3}=\frac{2}{(x-1)^3}.$$

12. Verify that the function $y=\dfrac{1}{x}$, $x>0$, satisfies the differential equation $x^3y''+2x^2y'=0$.

We calculate the first and second derivatives and substitute them into the differential equation.

$$y=x^{-1},\ y'=-x^{-2}=\frac{-1}{x^2},\text{ and } y''=2x^{-3}=\frac{2}{x^3},\text{ so we have } x^3y''+2x^2y'=x^3\left(\frac{2}{x^3}\right)+2x^2\left(\frac{-1}{x^2}\right)=2-2=0.$$

Lesson Ten

1. Use the chain rule to find the derivative of $f(x)=(4x-1)^3$.

$f'(x)=3(4x-1)^2\frac{d}{dx}[4x-1]=3(4x-1)^2(4)=12(4x-1)^2$.

2. Use the chain rule to find the derivative of $f(t)=\sqrt{5-t}$.

$\frac{d}{dt}\left[\sqrt{5-t}\right]=\frac{d}{dt}\left[(5-t)^{\frac{1}{2}}\right]=\frac{1}{2}(5-t)^{-\frac{1}{2}}(-1)=\frac{-1}{2\sqrt{5-t}}$.

3. Use the chain rule to find the derivative of $f(x)=\cos 4x$.

$f'(x)=-(\sin 4x)(4)=-4\sin 4x$.

4. Use the chain rule to find the derivative of $f(x)=5\tan 3x$.

$f'(x)=5(\sec^2 3x)(3)=15\sec^2 3x$.

5. Find the equation of the tangent line to the graph $f(x)=\left(9-x^2\right)^{\frac{2}{3}}$ at the point $(1,4)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

$f'(x)=\frac{2}{3}\left(9-x^2\right)^{-\frac{1}{3}}(-2x)=\frac{-4x}{3\left(9-x^2\right)^{1/3}}$, and $f'(1)=\frac{-4}{3(2)}=\frac{-2}{3}$. The equation of the tangent line is $y-4=\frac{-2}{3}(x-1)$, or $y=\frac{-2}{3}x+\frac{14}{3}$. If you graph the original function together with the tangent line, you will see that they intersect at the point $(1,4)$.

6. Find the equation of the tangent line to the graph $f(x)=26-\sec^3 4x$ at the point $(0,25)$. Use a graphing utility to graph the function and tangent line in the same viewing window.

$f'(x)=-3\sec^2 4x\frac{d}{dx}[\sec 4x]=-3\sec^2 4x\sec 4x\tan 4x(4)=-12\sec^3 4x\tan 4x$, and $f'(0)=0$. The equation of the tangent line is $y-25=0(x-0)$, or $y=25$. If you graph the original function together with the tangent line, you will see that they intersect at the point $(0,25)$. The graph has a horizontal tangent at $x=0$.

7. Find the derivative of $f(x)=\sin(\tan 2x)$.

$f'(x)=\cos(\tan 2x)\frac{d}{dx}[\tan 2x]=\cos(\tan 2x)(\sec^2 2x)(2)=2(\sec^2 2x)\cos(\tan 2x)$.

8. Find the derivative of $f(x)=\left(\frac{3x-1}{x^2+3}\right)^2$.

$$f'(x)=2\left(\frac{3x-1}{x^2+3}\right)\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]=2\left(\frac{3x-1}{x^2+3}\right)\frac{(x^2+3)(3)-(3x-1)(2x)}{\left(x^2+3\right)^2}$$
$$=\frac{-2(3x-1)(3x^2-2x-9)}{\left(x^2+3\right)^3}.$$

9. Find the derivative of the function $f(x)=\dfrac{5}{x^3-2}$ at the point $\left(-2,-\dfrac{1}{2}\right)$.

$f(x)=5\left(x^3-2\right)^{-1}$, and $f'(x)=-5\left(x^3-2\right)^{-2}\left(3x^2\right)=\dfrac{-15x^2}{\left(x^3-2\right)^2}$. Hence, $f'(-2)=\dfrac{-60}{100}=-\dfrac{3}{5}$.

10. Find the second derivative of $f(x)=\dfrac{4}{(x+2)^3}$.

$f(x)=4(x+2)^{-3}$, and $f'(x)=-12(x+2)^{-4}$. Hence $f''(x)=48(x+2)^{-5}=\dfrac{48}{(x+2)^5}$.

11. Find the derivative of $f(x)=\tan^2 x-\sec^2 x$. What do you observe?

$f'(x)=2\tan x\sec^2 x-2\sec x(\sec x\tan x)=0$. The derivative is zero because the trigonometric identity $\tan^2 x+1=\sec^2 x$ implies that the function f is constant.

Lesson Eleven

1. Find y' by implicit differentiation: $x^2 + y^2 = 9$.

Differentiating both sides with respect to the variable x, you obtain $2x + 2yy' = 0$, which implies that $2yy' = -2x$, or $y' = \frac{-x}{y}$.

2. Find y' by implicit differentiation: $x^3 - xy + y^2 = 7$.

We differentiate both sides with respect to x. Note that the second term requires the product rule.

$$\begin{aligned} 3x^2 - xy' - y + 2yy' &= 0 \\ (2y - x)y' &= y - 3x^2 \\ y' &= \frac{y - 3x^2}{2y - x} \end{aligned}$$

3. Find y' by implicit differentiation: $y = \sin xy$.

Note that the derivative of the right-hand side requires both the chain rule and the product rule.

$$\begin{aligned} y' &= \cos(xy)\frac{d}{dx}\left[xy\right] = \cos(xy)\left(xy' + y\right) = xy'\cos(xy) + y\cos(xy) \\ (1 - x\cos xy)y' &= y\cos xy \\ y' &= \frac{y\cos xy}{1 - x\cos xy} \end{aligned}$$

4. Find y' by implicit differentiation: $\tan(x + y) = x$.

$$\begin{aligned} \sec^2(x + y)(1 + y') &= 1 \\ 1 + y' &= \cos^2(x + y) \\ y' &= \cos^2(x + y) - 1 \end{aligned}$$

5. Find the slope of the tangent line to the graph $\left(4 - x\right)y^2 = x^3$ at the point $(2, 2)$.

We differentiate both sides implicitly and then substitute the given values for x and y.

$$\begin{aligned} (4 - x)2yy' - y^2 &= 3x^2 \\ (4 - 2)2(2)y' - (2)^2 &= 3(2)^2 \\ 8y' &= 12 + 4 \\ y' &= \frac{16}{8} = 2 \end{aligned}$$

6. Find an equation of the tangent line to the graph of $(x + 2)^2 + \left(y - 3\right)^2 = 37$ at the point $\left(4, 4\right)$.

$$\begin{aligned} 2(x + 2) + 2(y - 3)y' &= 0 \\ (y - 3)y' &= -(x + 2) \\ y' &= -\frac{x + 2}{y - 3} \end{aligned}$$

At the point (4, 4), $y' = -\frac{4 + 2}{4 - 3} = -6$. The equation of the tangent line is $y - 4 = -6(x - 4)$, or $y = -6x + 28$.

7. Find the second derivative y'' if $y^2 = x^3$.

$2yy' = 3x^2$, and $y' = \dfrac{3x^2}{2y}$. We now differentiate implicitly again, which gives

$$y'' = \frac{2y(6x) - 3x^2(2y')}{(2y)^2} = \frac{6xy - 3x^2y'}{2y^2}$$. Now replace y' to obtain

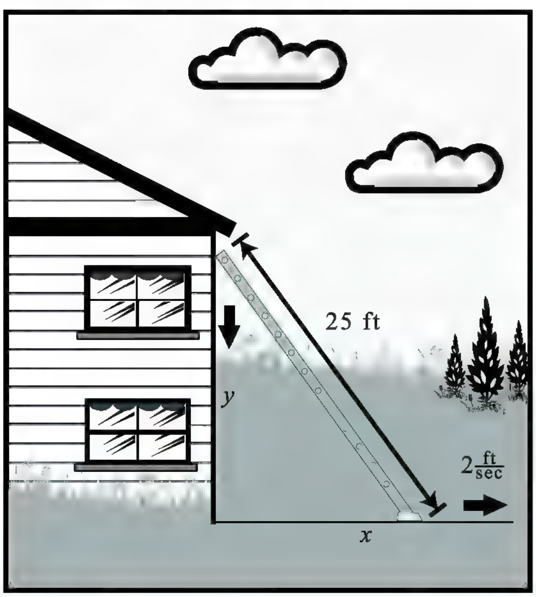
$$y'' = \frac{6xy - 3x^2\left(\dfrac{3x^2}{2y}\right)}{2y^2} = \frac{12xy^2 - 9x^4}{4y^3}.$$

8. Air is being pumped into a spherical balloon at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

The formula relating the variables is the volume of a sphere, $V = \frac{4}{3}\pi r^3$. Differentiating with respect to time t, you obtain $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$. Solving for the rate of change of the radius and substituting in the known quantities, you have

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi(2)^2}(4.5) = \frac{9}{32\pi} \approx 0.0895 \text{ feet per second.}$$

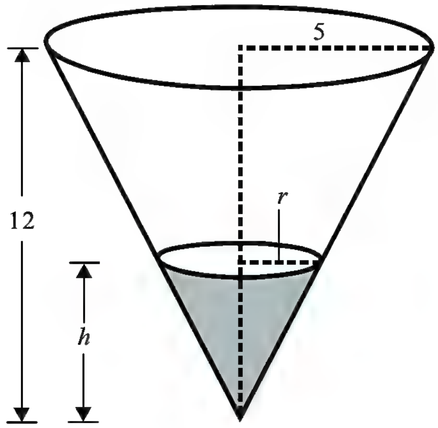
9. A 25-foot ladder is leaning against the wall of a house. The base of the ladder is being pulled away from the wall at a rate of 2 feet per second. How fast is the top of the ladder moving down the wall when the base is 7 feet from the wall?



Let x be the horizontal distance from the wall to the base of the ladder, and let y be the vertical distance from the top of the ladder to the ground. Both x and y depend on time, t. From the Pythagorean theorem, $x^2 + y^2 = 25$. Differentiating with respect to time (not x), $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \Rightarrow \dfrac{dy}{dt} = \dfrac{-x}{y}\dfrac{dx}{dt}$.

We know that $\dfrac{dx}{dt} = 2$. When $x = 7,\ y = 24$, so we have $\dfrac{dy}{dt} = \dfrac{-7}{24}(2) = -\dfrac{7}{12}$ feet per second.

10. A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.



The radius of the tank is 5, and the height is 12. Let h be the depth of the water and r be the radius. By similar triangles, $\frac{r}{5}=\frac{h}{12} \Rightarrow r=\frac{5h}{12}$. We know that $\frac{dV}{dt}=10$, and we want to find $\frac{dh}{dt}$. The volume of a cone is $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h=\frac{25\pi h^3}{3(144)}$. Differentiating with respect to time, $\frac{dV}{dt}=\frac{25\pi}{3(144)}\left(3h^2\frac{dh}{dt}\right) \Rightarrow \frac{dh}{dt}=\frac{144}{25\pi h^2}\frac{dV}{dt}$. When $h=8$, $\frac{dh}{dt}=\frac{144}{25\pi 8^2}(10)=\frac{9}{10\pi}$ feet per minute.

Lesson Twelve

1. Find the critical numbers of the function $f(x) = x^4 - 4x^2$.

   The function is differentiable everywhere, so the critical numbers are the x-values where the derivative is zero. We see that $f'(x) = 4x^3 - 8x = 4x(x^2 - 2) = 0$, and hence the 3 critical numbers are 0 and $\pm\sqrt{2}$.

2. Find the critical numbers of the function $h(x) = \sin^2 x + \cos x,\ 0 < x < 2\pi$.

   The function is differentiable everywhere, so the critical numbers are the x-values where the derivative is zero. We see that $h'(x) = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1) = 0$. In the given interval, $\sin x = 0$ when $x = \pi$. Similarly, $\cos x = \frac{1}{2}$ when $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Therefore there are 3 critical numbers: π, $\frac{\pi}{3}$, and $\frac{5\pi}{3}$.

3. Find the value of the derivative of the function $f(x) = \dfrac{x^2}{x^2 + 4}$ at its relative minimum $(0,0)$.

   $$f'(x) = \frac{(x^2+4)(2x) - x^2(2x)}{(x^2+4)^2} = \frac{8x}{(x^2+4)^2},\ \text{and } f'(0) = 0.$$

4. Find the value of the derivative of the function $f(x) = 4 - |x|$ at its relative maximum $(0,4)$.

   The derivative does not exist at $x = 0$. There is a sharp corner there.

5. Find the absolute extrema of the function $y = -x^2 + 3x - 5,\ [-2,1]$.

   We first find the critical numbers on the open interval: $y' = -2x + 3 = 0$ implies that $x = \frac{3}{2}$. There are no critical numbers in the interval, so we just evaluate the function at the endpoints. We see that $f(-2) = -15$, and $f(1) = -3$. The maximum value is -3, and the minimum value is -15.

6. Find the absolute extrema of the function $y = x^3 - \frac{3}{2}x^2,\ [-1,2]$.

   We see that $f'(x) = 3x^2 - 3x = 3x(x-1) = 0$. The critical numbers on the open interval are 0 and 1. We evaluate the function at these critical numbers and at the endpoints:

   $$f(0) = 0,\ f(1) = -\frac{1}{2},\ f(-1) = -\frac{5}{2},\ \text{and } f(2) = 2.$$

   The maximum value is 2, and the minimum value is $\frac{-5}{2}$.

7. Find the absolute extrema of the function $f(t) = 3 - |t - 3|,\ [-1,5]$.

   You see from the absolute value function that the only critical number is $t = 3$. Evaluating the function at this number and the endpoints gives $f(3) = 3 - 0 = 3,\ f(-1) = 3 - 4 = -1,$ and $f(5) = 3 - 2 = 1$. The maximum value is 3, and the minimum value is -1.

8. Find the absolute extrema of the function $y = 3x^{\frac{2}{3}} - 2x,\ [-1,1]$.

   We see that $y' = 2x^{-\frac{1}{3}} - 2 = \frac{2}{x^{1/3}} - 2 = 2\frac{1 - x^{1/3}}{x^{1/3}}$. We determine that 1 is a critical number because the derivative is 0 at $x = 1$ and that 0 is a critical number because the derivative is undefined at $x = 0$. We have $f(1) = 3 - 2 = 1,\ f(0) = 0,$ and $f(-1) = 3 + 2 = 5$. The maximum value is 5, and the minimum value is 0.

9. Locate the absolute extrema of the function $f(x)=2x-3$ (if any exist) on the following intervals:

a. $[0,2]$

b. $[0,2)$

c. $(0,2]$

d. $(0,2)$

The graph of f is a line of slope 2.

a. In this closed interval, the maximum value is $f(2)=2(2)-3=1$, and the minimum value is $f(0)=-3$.

b. The minimum value is $f(0)=-3$, but there is no maximum value.

c. The maximum value is $f(2)=1$, but there is no minimum value.

d. There are no relative extrema on this open interval.

10. True or false: The maximum of a function that is continuous on a closed interval can occur at 2 different values in the interval.

True. For example, the maximum value of $f(x)=x^2,\ -2\le x\le 2,$ occurs at the 2 endpoints, $f(2)=f(-2)=4$.

Lesson Thirteen

1. Identify the open intervals on which the function $f(x) = x^2 - 2x - 8$ is increasing or decreasing.

First calculate the critical numbers: $f'(x) = 2x - 2 = 0 \Rightarrow x = 1$. Hence we have to analyze the sign of the derivative on 2 intervals. On the interval $(-\infty, 1)$, $f'(x) < 0$, whereas on the interval $(1, \infty)$, $f'(x) > 0$. Hence the function is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.

2. Identify the open intervals on which the function $g(x) = x\sqrt{16 - x^2}$ is increasing or decreasing.

The domain of the function is $-4 \le x \le 4$. We find the critical numbers in the open interval $(-4, 4)$:

$$\begin{aligned} g(x) = x\left(16 - x^2\right)^{\frac{1}{2}} \Rightarrow g'(x) &= \frac{x}{2}\left(16 - x^2\right)^{-\frac{1}{2}}(-2x) + \left(16 - x^2\right)^{\frac{1}{2}} \\ &= -x^2\left(16 - x^2\right)^{-\frac{1}{2}} + \left(16 - x^2\right)^{\frac{1}{2}} \\ &= \left(16 - x^2\right)^{-\frac{1}{2}}\left(-x^2 + 16 - x^2\right) \\ &= \frac{2\left(8 - x^2\right)}{\left(16 - x^2\right)^{\frac{1}{2}}}. \end{aligned}$$

Setting $g'(x) = 0$, the critical numbers are $\pm\sqrt{8} = \pm 2\sqrt{2}$. On the interval $\left(-4, -2\sqrt{2}\right)$, $g'(x) < 0$; on the interval $\left(-2\sqrt{2}, 2\sqrt{2}\right)$, $g'(x) > 0$; and on the interval $\left(2\sqrt{2}, 4\right)$, $g'(x) < 0$. Therefore, g is increasing on the interval $\left(-2\sqrt{2}, 2\sqrt{2}\right)$ and decreasing on $\left(-4, -2\sqrt{2}\right)$ and $\left(2\sqrt{2}, 4\right)$.

3. **a.** Find the critical numbers of $f(x) = 2x^3 + 3x^2 - 12x$ (if any).

b. Find the open interval(s) on which the function is increasing or decreasing.

c. Apply the first derivative test to identify all relative extrema.

d. Use a graphing utility to confirm your results.

a. $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$. The critical numbers are -2 and 1.

b. f is increasing on $(-\infty, -2)$ and $(1, \infty)$ and decreasing on $(-2, 1)$.

c. By the first derivative test, there is a relative maximum at $(-2, 20)$ and a relative minimum at $(1, -7)$.

d. Graphing the original function with your graphing utility should confirm these conclusions. Hint: Consider the viewing window $[-3, 3] \times [-7, 20]$ as a starting point for your exploration.

4. **a.** Find the critical numbers of $f(x)=\dfrac{x^2}{x^2-9}$ (if any).

b. Find the open interval(s) on which the function is increasing or decreasing.

c. Apply the first derivative test to identify all relative extrema.

d. Use a graphing utility to confirm your results.

a. $f'(x)=\dfrac{(x^2-9)(2x)-x^2(2x)}{(x^2-9)^2}=\dfrac{-18x}{(x^2-9)^2}$. The critical number is 0.

b. We analyze the sign of the derivative on the 4 intervals determined by the critical number 0 and the 2 vertical asymptotes, $x=\pm 3$. On the interval $(-\infty,-3)$, $f'>0$, increasing; on $(-3,0)$, $f'>0$, increasing; on $(0,3)$, $f'<0$, decreasing; and on $(3,\infty)$, $f'<0$, decreasing.

c. By the first derivative test, there is a relative maximum at (0,0).

d. Graphing the original function with your graphing utility should confirm these conclusions. Hint: Consider the viewing window $[-5,5]\times[-10,10]$.

5. The electric power P in watts in a direct-current circuit with 2 resistors, R_1 and R_2, connected in parallel is $P=\dfrac{vR_1R_2}{(R_1+R_2)^2}$, where v is the voltage. If v and R_1 are held constant, what resistance R_2 produces maximum power?

Observe that R_1 and R_2 are positive. We differentiate with respect to R_2, holding v and R_1 constant:

$$\begin{aligned}\frac{dP}{dR_2}&=\frac{(R_1+R_2)^2(vR_1)-vR_1R_2\left[2(R_1+R_2)(1)\right]}{(R_1+R_2)^4}\\&=\frac{(vR_1)(R_1-R_2)}{(R_1+R_2)^3}\\&=0\Rightarrow R_1=R_2.\end{aligned}$$

By the first derivative test, this is a maximum.

6. Explain why Rolle's theorem does not apply to the function $f(x)=1-|x-1|$ on the interval $[0,2]$.

The function is not differentiable on the interval $(0,2)$ because the derivative does not exist at 1.

7. Determine whether the mean value theorem can be applied to $f(x)=\sin x$ on the closed interval $[0,\pi]$. If the mean value theorem can be applied, find all values of c in the interval $(0,\pi)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

The function is continuous on the closed interval and differentiable on the open interval, so the mean value theorem applies. $f'(x)=\cos x$, so we have

$$f'(c)=\frac{f(b)-f(a)}{b-a}\Rightarrow \cos c=\frac{\sin\pi-\sin 0}{\pi-0}=0.$$

On the given interval, $c=\dfrac{\pi}{2}$.

8. True or false? The sum of 2 increasing functions is increasing.

True. Let $h(x) = f(x) + g(x)$, where f and g are increasing. Then $h'(x) = f'(x) + g'(x) > 0$.

9. True or false? There is a relative maximum or minimum at each critical number.

False. Consider $f(x) = x^3$. It has a critical number at $x = 0$, but this is not a relative extremum.

10. The function $s(t) = t^2 - 7t + 10$ describes the motion of a particle along a line.

a. Find the velocity function of the particle at any time $t \geq 0$.

b. Identify the time interval(s) in which the particle is moving in a positive direction.

c. Identify the time interval(s) in which the particle is moving in a negative direction.

d. Identify the time(s) at which the particle changes direction.

a. $v(t) = s'(t) = 2t - 7$.

b. $v(t) = 0$ when $t = \frac{7}{2}$. The particle is moving in the positive direction for $t > \frac{7}{2}$ because the derivative is positive on that interval.

c. The particle is moving in the negative direction for $0 < t < \frac{7}{2}$.

d. The particle changes direction at $t = \frac{7}{2}$.

Lesson Fourteen

1. Determine the open intervals on which the function $f(x) = x^2 - x - 2$ is concave upward or concave downward.

$f'(x) = 2x - 1$, and $f''(x) = 2 > 0$. The function is concave upward on $(-\infty, \infty)$.

2. Determine the open intervals on which the function $f(x) = 3x^2 - x^3$ is concave upward or concave downward.

$f'(x) = 6x - 3x^2$, and $f''(x) = 6 - 6x$. We determine the intervals where the second derivative is positive as well as those where it is negative. We conclude that the function is concave upward on $(-\infty, 1)$ and concave downward on $(1, \infty)$.

3. Determine the open intervals on which the function $f(x) = 2x - \tan x$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, is concave upward or concave downward.

$f'(x) = 2 - \sec^2 x$, and $f''(x) = -2\sec^2 x \tan x$. $f''(x) = 0$ at $x = 0$. The function is concave upward on $\left(\frac{-\pi}{2}, 0\right)$ and concave downward on $\left(0, \frac{\pi}{2}\right)$.

4. Find the points of inflection and discuss the concavity of the graph of $f(x) = -x^4 + 24x^2$.

$f'(x) = -4x^3 + 48x$, and $f''(x) = -12x^2 + 48 = 12(4 - x^2) = 12(2 + x)(2 - x)$.

The function is concave upward on $(-2, 2)$ and concave downward on $(-\infty, -2)$, $(2, \infty)$. Points of inflection are at $(-2, 80)$ and $(2, 80)$. Note that the function is even.

5. Find the points of inflection and discuss the concavity of the graph of $f(x) = x(x-4)^3$.

$$f'(x) = x(3(x-4)^2) + (x-4)^3 = (x-4)^2(4x-4)$$
$$f''(x) = (x-4)^2(4) + (4x-4)2(x-4) = (x-4)(4x-16+8x-8) = 12(x-4)(x-2)$$

$f'' = 0$ when $x = 2,\ 4$. The function is concave upward on $(-\infty, 2)$ and $(4, \infty)$ and concave downward on $(2, 4)$. There are points of inflection at $(2, -16)$ and $(4, 0)$.

6. Find the points of inflection and discuss the concavity of the graph of $f(x) = \sin x + \cos x$, $[0, 2\pi]$.

$f'(x) = \cos x - \sin x$, and $f''(x) = -\sin x - \cos x$. $f'' = 0$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. The function is concave upward on $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ and concave downward on $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$. There are points of inflection at $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$.

7. Find all relative extrema of $f(x) = x^3 - 3x^2 + 3$. Use the second derivative test where applicable.

$f'(x) = 3x^2 - 6x = 3x(x-2)$, so the critical numbers are 0 and 2. We evaluate the second derivative at these points: $f''(x) = 6x - 6$, so $f''(0) < 0$, and $f''(2) > 0$. Therefore $(0, 3)$ is a relative maximum, and $(2, -1)$ is a relative minimum.

8. Find all relative extrema of $f(x) = x + \frac{4}{x}$. Use the second derivative test where applicable.

$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = 0 \Rightarrow x = \pm 2$ are the critical numbers. Note that 0 is not a critical number because it is not in the domain of the function. $f''(x) = \frac{8}{x^3}$, so $f''(-2) < 0$, and $f''(2) > 0$. Therefore, $(-2, -4)$ is a relative maximum, and $(2, 4)$ is a relative minimum.

9. Discuss the concavity of the cube root function $f(x) = x^{\frac{1}{3}}$.

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, and $f''(x) = \frac{-2}{9}x^{-\frac{2}{3}}$. The function is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$. Note that 0 is a critical number. Try verifying this with a graphing utility.

10. Discuss the concavity of the function $g(x) = x^{\frac{2}{3}}$.

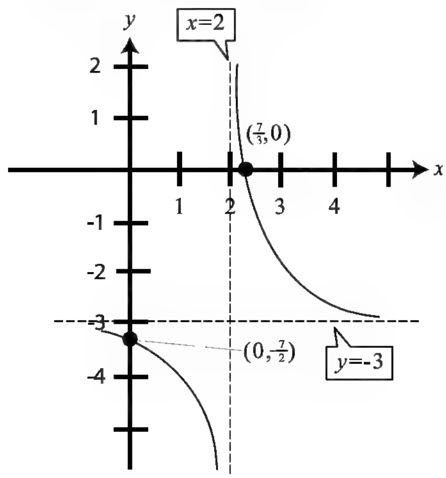
$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$, and $f''(x) = \frac{-2}{9}x^{-\frac{4}{3}}$. You see that the function is concave downward on the intervals $(-\infty, 0)$ and $(0, \infty)$. Notice the cusp at the origin.

Lesson Fifteen

1. Analyze and sketch the graph of the function $f(x) = \dfrac{1}{x-2} - 3$.

$$f(x) = \frac{1}{x-2} - 3,\ f'(x) = \frac{-1}{(x-2)^2},\ \text{and}\ f''(x) = \frac{2}{(x-2)^3}$$

There are no critical numbers, relative extrema, or points of inflection. There are a horizontal asymptote, $y = -3$, and a vertical asymptote, $x = 2$. In fact, the graph of f is that of $y = \dfrac{1}{x}$ shifted 2 units to the right and 3 units downward.

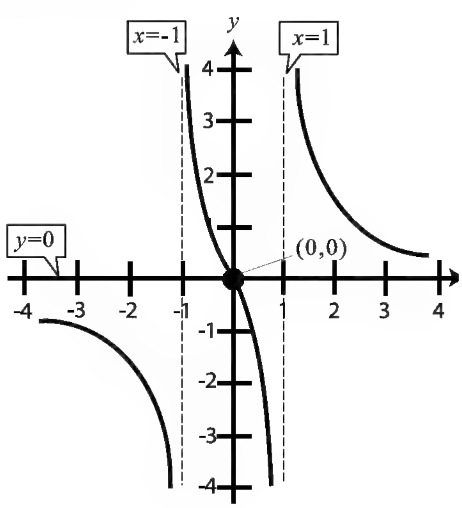


2. Analyze and sketch the graph of the function $f(x) = \dfrac{3x}{x^2 - 1}$.

$$f(x) = \frac{3x}{x^2-1},\ f'(x) = \frac{-3(x^2+1)}{\left(x^2-1\right)^2},\ \text{and}\ f''(x) = \frac{6x(x^2+3)}{\left(x^2-1\right)^3}.$$

There are vertical asymptotes at $x = \pm 1$ and a horizontal asymptote at $y = 0$.

There are no critical numbers, but $(0,0)$ is a point of inflection. There is symmetry with respect to the origin.

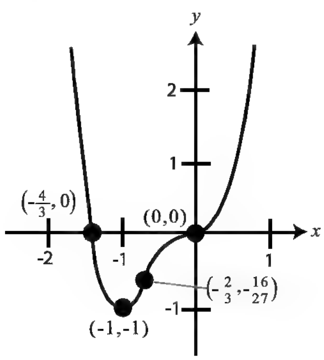


3. Analyze and sketch the graph of the function $f(x)=3x^4+4x^3$.

$f(x)=3x^4+4x^3=x^3(3x+4)$, $f'(x)=12x^3+12x^2=12x^2(x+1)$, and $f''(x)=36x^2+24x=12x(3x+2)$.

The intercepts are $(0,0)$ and $\left(\frac{-4}{3},0\right)$. Critical numbers are 0 and -1. There is a relative minimum at $(-1,-1)$. There are inflection points at $(0,0)$ and $\left(\frac{-2}{3},\frac{-16}{27}\right)$. The graph is increasing on $(-1,\infty)$ and decreasing on $(-\infty,-1)$.

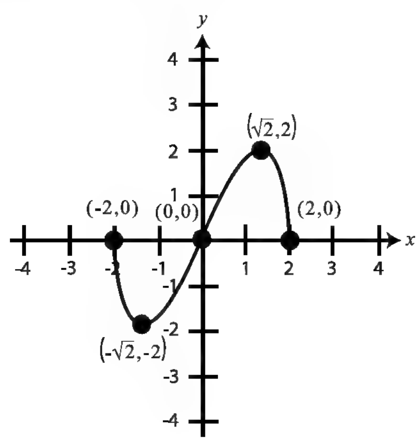
The graph is concave upward on $\left(-\infty,-\frac{2}{3}\right)$ and $(0,\infty)$ and concave downward on $\left(-\frac{2}{3},0\right)$.



4. Analyze and sketch the graph of the function $f(x)=x\sqrt{4-x^2}$.

$f(x)=x\sqrt{4-x^2}$, $f'(x)=\dfrac{4-2x^2}{\sqrt{4-x^2}}$, and $f''(x)=\dfrac{2x(x^2-6)}{(4-x^2)^{\frac{3}{2}}}$.

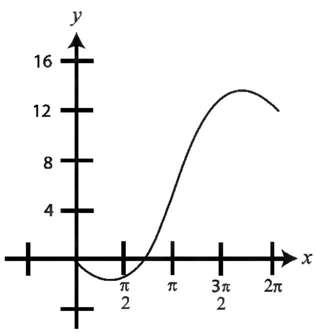
The domain is $-2\le x\le 2$, and the intercepts are $(\pm 2,0)$ and $(0,0)$. The critical numbers are $\pm\sqrt{2}$. There is a relative minimum at $(-\sqrt{2},-2)$ and a relative maximum at $(\sqrt{2},2)$. The graph is symmetric about the origin, and there is a point of inflection at the origin.



5. Analyze and sketch the graph of the function $f(x) = 2x - 4\sin x,\ 0 \le x \le 2\pi$.

$f(x) = 2x - 4\sin x$, $f'(x) = 2 - 4\cos x$, and $f''(x) = 4\sin x$.

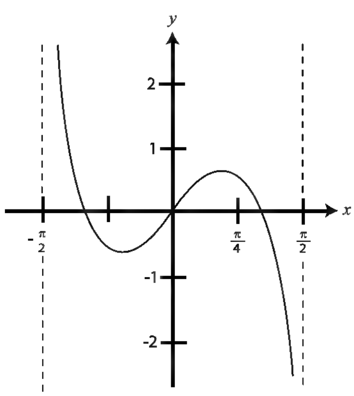
The critical numbers are solutions to $\cos x = \frac{1}{2}$, or $x = \frac{\pi}{3}, \frac{5\pi}{3}$. There is a relative minimum at $\left(\frac{\pi}{3}, \frac{2\pi}{3} - 2\sqrt{3}\right)$ and a relative maximum at $\left(\frac{5\pi}{3}, \frac{10\pi}{3} + 2\sqrt{3}\right)$. There is a point of inflection at $(\pi, 2\pi)$.



6. Analyze and sketch the graph of the function $f(x) = 2x - \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

$f(x) = 2x - \tan x$, $f'(x) = 2 - \sec^2 x$, and $f''(x) = -2\sec^2 x \tan x$.

The critical numbers are solutions to $\sec^2 x = 2$, or $\cos^2 x = \frac{1}{2}$, which gives $x = \pm\frac{\pi}{4}$. There is a relative maximum at $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$ and a relative minimum at $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$. There is a point of inflection at the origin, and there are vertical asymptotes at $x = \pm\frac{\pi}{2}$. The function is symmetric about the origin.



7. Find the tangent line approximation $T(x)$ to $f(x) = x^5$ at $(2, 32)$. Compare the values of T and f for $x = 1.9$ and $x = 2.01$.

$f(x) = x^5$, so $f'(x) = 5x^4$. The tangent line at $(2, 32)$ is $y - f(2) = f'(2)(x - 2)$, which gives $y - 32 = 80(x - 2)$, or $T(x) = 80x - 128$. Comparing values, $T(1.9) = 24$, $f(1.9) \approx 24.7610$, $T(2.01) = 32.8$, and $f(2.01) \approx 32.8080$.

8. Find the tangent line approximation $T(x)$ to $f(x) = \sin x$ at $(2, \sin 2)$. Compare the values of T and f for $x = 1.9$ and $x = 2.01$.

$f(x) = \sin x$, so $f'(x) = \cos x$. The tangent line at $(2, \sin 2)$ is $y - f(2) = f'(2)(x-2)$, which gives $y - \sin 2 = (\cos 2)(x-2)$, or $T(x) = (\cos 2)x + \sin 2 - 2\cos 2$. Comparing values, $T(1.9) \approx 0.9509$, $f(1.9) \approx 0.9463$, $T(2.01) \approx 0.9051$, and $f(2.01) \approx 0.9051$.

9. Find the differential dy for $y = 3x^2 - 4$.

$\frac{dy}{dx} = \frac{d}{dx}\left[3x^2 - 4\right] = 6x$, which implies that $dy = 6x\,dx$.

10. Find the differential dy for $y = x\cos x$.

$\frac{dy}{dx} = \frac{d}{dx}\left[x\cos x\right] = -x\sin x + \cos x$, which implies that $dy = (-x\sin x + \cos x)\,dx$.

Lesson Sixteen

1. Find 2 positive numbers such that their product is 185 and their sum is a minimum.

Let x and y be the 2 positive numbers such that $xy = 185$, so $y = \frac{185}{x}$. Their sum is $S = x + y = x + \frac{185}{x}$, and $S' = 1 - \frac{185}{x^2} = 0$ when $x = \sqrt{185}$. $S'' = \frac{370}{x^3} > 0$, so the second derivative test confirms that the sum is a minimum when $x = y = \sqrt{185}$.

2. Find 2 positive numbers such that the second number is the reciprocal of the first number and their sum is a minimum.

Let x be a positive number. So $y = \frac{1}{x}$, $S = x + y = x + \frac{1}{x}$, and $\frac{dS}{dx} = 1 - \frac{1}{x^2} = 0$ when $x = 1$. $S'' = \frac{2}{x^3} > 0$, so the second derivative test confirms the sum is a minimum when $x = 1$ and $\frac{1}{x} = 1$. Thus the 2 numbers are $x = 1$ and $y = 1$.

3. Find the point on the graph of $f(x) = x^2$ that is closest to the point $\left(2, \frac{1}{2}\right)$.

Let $\left(x, x^2\right)$ be a point on the parabola. The distance between the points is

$$d = \sqrt{(x-2)^2 + \left(x^2 - \frac{1}{2}\right)^2} = \sqrt{x^4 - 4x + \frac{17}{4}}.$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 4x + \frac{17}{4}$. We have $f'(x) = 4x^3 - 4 = 4(x^3 - 1)$, so the only critical number is 1. By the first derivative test, the point nearest $\left(2, \frac{1}{2}\right)$ is (1,1).

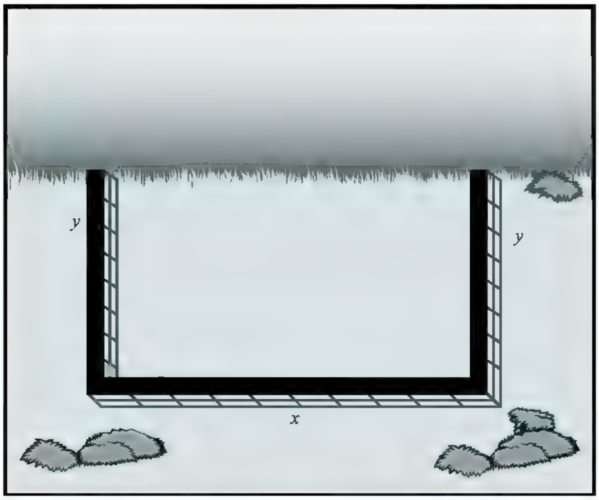
4. Find the point on the graph of $f(x) = \sqrt{x}$ that is closest to the point $(4,0)$.

Let $\left(x, \sqrt{x}\right)$ be a point on the graph. The distance between the points is

$$d = \sqrt{(x-4)^2 + \left(\sqrt{x} - 0\right)^2} = \sqrt{x^2 - 7x + 16}.$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^2 - 7x + 16$. We have $f'(x) = 2x - 7$, so the only critical number is $x = \frac{7}{2}$. By the first derivative test, the point nearest $(4,0)$ is $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$.

5. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 square meters to provide enough grass for the herd. What dimensions will require the least amount of fencing if no fencing is needed along the river?



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We are given the area: $A = xy = 245{,}000$. And $S = x + 2y = x + \dfrac{490{,}000}{x}$, where S is the length of the fence needed. Thus $S' = 1 - \dfrac{490{,}000}{x^2} = 0$ when $x = 700$. We see that $S'' = \dfrac{980{,}000}{x^3} > 0$, so the second derivative test confirms that S is a minimum when $x = 700$ and $y = 350$.

6. The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and the square that produce a minimum total area.

Let x be the length of the side of the square and y be the length of a side of the triangle. We see that $4x + 3y = 10$, which gives $x = \dfrac{10-3y}{4}$. The sum of the areas of the 2 figures is

$A = x^2 + \dfrac{1}{2}y\left(\dfrac{\sqrt{3}}{2}y\right) = \dfrac{(10-3y)^2}{16} + \dfrac{\sqrt{3}}{4}y^2$. And $\dfrac{dA}{dy} = \dfrac{1}{8}(10-3y)(-3) + \dfrac{\sqrt{3}}{2}y = 0$.

Solving for y, $-30 + 9y + 4\sqrt{3}y = 0 \Rightarrow y = \dfrac{30}{9+4\sqrt{3}}$. And $\dfrac{d^2A}{dy^2} = \dfrac{9+4\sqrt{3}}{8} > 0$, so the total area is a minimum when $y = \dfrac{30}{9+4\sqrt{3}}$ and $x = \dfrac{10\sqrt{3}}{9+4\sqrt{3}}$.

Lesson Seventeen

1. A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

Let the printed area be $xy = 30$, or $y = \dfrac{30}{x}$. Since the margins are 1 inch on a side, the amount of paper used is $A = (x+2)(y+2) = (x+2)\left(\dfrac{30}{x}+2\right)$. Therefore, $A' = (x+2)\left(\dfrac{-30}{x^2}\right)+\left(\dfrac{30}{x}+2\right) = \dfrac{2\left(x^2-30\right)}{x^2} = 0$ when $x = \sqrt{30}$. By the first derivative test, this value yields a minimum. Hence $y = \sqrt{30}$ as well, and the dimensions are those of a square: $2+\sqrt{30}$ by $2+\sqrt{30}$ inches.

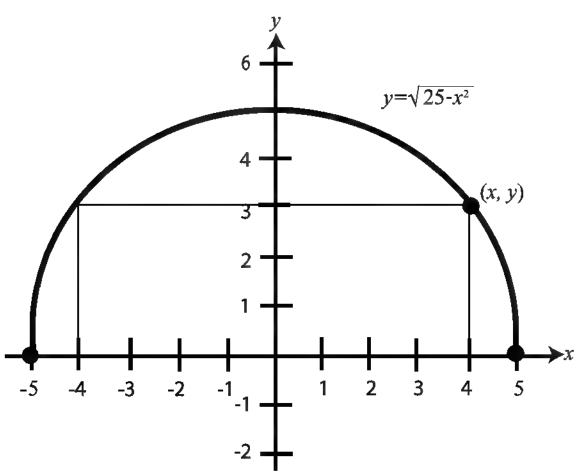
2. A rectangular page is to contain 36 square inches of print. The margins on each side are 1.5 inches. Find the dimensions of the page such that the least amount of paper is used.

Let the printed area be $xy = 36$. Since the margins are 1.5 inches on a side, the amount of paper used is $A = (x+3)(y+3) = (x+3)\left(\dfrac{36}{x}+3\right) = 36+\dfrac{108}{x}+3x+9$. Thus $A' = \dfrac{-108}{x^2}+3 = 0$ when $3x^2 = 108$, or $x = 6$. By the first derivative test, this value yields a minimum. Hence, $y = 6$ as well, and the dimensions are those of a square: 9 by 9 inches.

3. In an autocatalytic reaction, the product formed is a catalyst for the reaction. If Q_0 is the amount of the original substance and x is the amount of catalyst formed, the rate of the chemical reaction is $\dfrac{dQ}{dx} = kx\left(Q_0 - x\right)$. For what value of x will the rate of chemical reaction be greatest?

$\dfrac{dQ}{dx} = kx\left(Q_0 - x\right) = kQ_0x - kx^2$, and $\dfrac{d^2Q}{dQ^2} = kQ_0 - 2kx = 0$ when $x = \dfrac{Q_0}{2}$. The third derivative is negative, so the rate is a maximum at $x = \dfrac{Q_0}{2}$.

4. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 5.

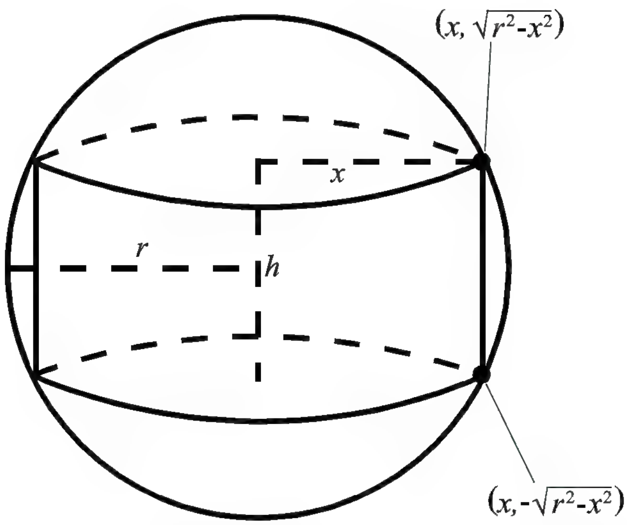


$A = 2xy = 2x\sqrt{25-x^2}$, and $A' = \dfrac{2(25-2x^2)}{\sqrt{25-x^2}} = 0$ when $x = \dfrac{5}{\sqrt{2}}$. By the first derivative test, this is a maximum. The largest rectangle has dimensions $2x = 5\sqrt{2}$ by $y = \dfrac{5}{\sqrt{2}}$.

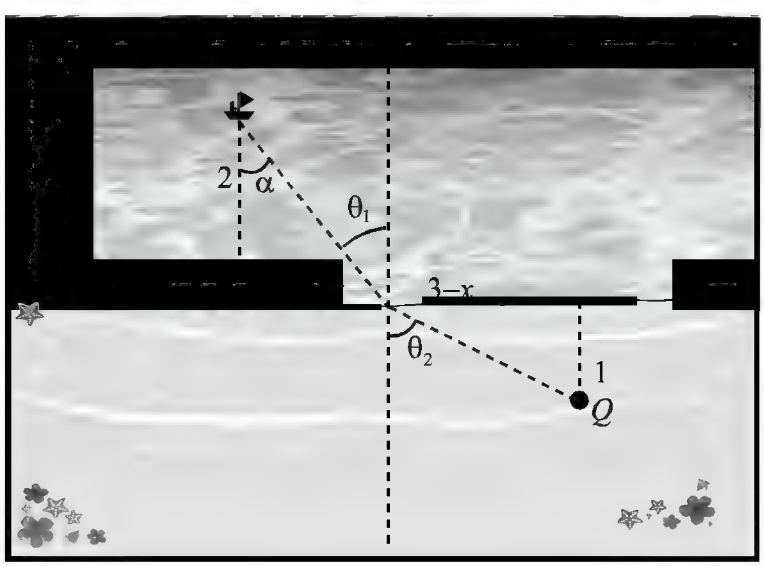
5. Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

Let x be the radius of the cylinder and $V = \pi x^2 h$ be its volume. From the figure, you see that $h = 2\sqrt{r^2 - x^2}$, so $V = 2\pi x^2\sqrt{r^2 - x^2}$. The derivative simplifies to $V' = \dfrac{2\pi x}{\sqrt{r^2 - x^2}}\left(2r^2 - 3x^2\right)$, and $V' = 0$ when $x = \dfrac{\sqrt{6}\,r}{3}$. By the first derivative test, this is a maximum. We have $h = \dfrac{2r}{\sqrt{3}}$ and

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$



6. A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland. He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row to reach point Q in the least time?



The distance rowed is $\sqrt{x^2+4}$, and the distance walked is $\sqrt{1+(3-x)^2}$. The total time is

$T=\dfrac{\sqrt{x^2+4}}{2}+\dfrac{\sqrt{1+(3-x)^2}}{4}=\dfrac{\sqrt{x^2+4}}{2}+\dfrac{\sqrt{x^2-6x+10}}{4}$. Taking the derivative,

$T'=\dfrac{x}{2\sqrt{x^2+4}}+\dfrac{x-3}{4\sqrt{x^2-6x+10}}$. Set the derivative equal to 0 and solve for x.

$$\frac{x}{2\sqrt{x^2+4}}=-\frac{x-3}{4\sqrt{x^2-6x+10}}$$

$$\frac{x^2}{x^2+4}=\frac{x^2-6x+9}{4(x^2-6x+10)}$$

$$x^4-6x^3+9x^2+8x-12=0$$

Using a graphing utility, you find there is only 1 root, $x=1$, on the interval $[0,3]$. Checking the endpoints 0 and 3, you see that $x=1$ yields the minimum time. So he should row to a point 1 mile from the nearest point on the coast.

Lesson Eighteen

1. Find the integral: $\int (x+7)\,dx$.

$$\int (x+7)\,dx = \int x\,dx + \int 7\,dx = \frac{x^2}{2} + 7x + C.$$

2. Find the integral: $\int (2x - 3x^2)\,dx$.

$$\int (2x-3x^2)\,dx = 2\frac{x^2}{2} - 3\frac{x^3}{3} + C = x^2 - x^3 + C.$$

3. Find the integral: $\int \frac{x+6}{\sqrt{x}}\,dx$.

$$\int \frac{x+6}{\sqrt{x}}\,dx = \int \left(x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}\right)dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 6\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}x^{\frac{3}{2}} + 12\sqrt{x} + C.$$

4. Find the integral: $\int \frac{x^2+2x-3}{x^4}\,dx$.

$$\int \frac{x^2+2x-3}{x^4}\,dx = \int \left(x^{-2} + 2x^{-3} - 3x^{-4}\right) = \frac{x^{-1}}{-1} + 2\frac{x^{-2}}{-2} - 3\frac{x^{-3}}{-3} + C = \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C.$$

5. Find the integral: $\int (5\cos x + 4\sin x)\,dx$.

$$\int (5\cos x + 4\sin x)\,dx = 5\sin x - 4\cos x + C.$$

6. Find the integral: $\int \left(\theta^2 + \sec^2\theta\right)d\theta$.

$$\int \left(\theta^2 + \sec^2\theta\right)d\theta = \frac{\theta^3}{3} + \tan\theta + C.$$

7. Solve the differential equation $f'(x) = 6x,\ f(0) = 8$.

$f(x) = \int 6x\,dx = 3x^2 + C$. Now use the initial condition: $8 = f(0) = 6(0) + C \Rightarrow C = 8$. The final answer is $f(x) = 3x^2 + 8$.

8. Solve the differential equation $h'(t) = 8t^3 + 5,\ h(1) = -4$.

$h(t) = \int (8t^3 + 5)\,dt = 2t^4 + 5t + C$. Now use the initial condition:

$-4 = h(1) = 2(1) + 5(1) + C \Rightarrow C = -11$. The final answer is $h(t) = 2t^4 + 5t - 11$.

9. A ball is thrown vertically from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?

The position function is $s(t) = \frac{1}{2}gt^2 + v_0t + s_0 = -16t^2 + 60t + 6$. The velocity is $v(t) = s'(t) = -32t + 60$.

Setting the velocity equal to 0, you have $32t = 60$, so $t = \frac{15}{8}$. The maximum height is

$$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 = \frac{249}{4} = 62.25 \text{ feet.}$$

10. With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?

We are asked to find the initial velocity v_0 needed to propel the object to 550 feet. The position function is $s(t)=\frac{1}{2}gt^2+v_0t+s_0=-16t^2+v_0t$, and $v(t)=-32t+v_0$. So $v(t)=0 \Rightarrow t=\frac{v_0}{32}$. Thus we have

$550=-16\left(\frac{v_0}{32}\right)^2+v_0\left(\frac{v_0}{32}\right)$, which we solve for the initial velocity:

$550=\frac{-16v_0^{\ 2}}{32^2}+\frac{v_0^{\ 2}}{32}=\frac{v_0^{\ 2}}{64} \Rightarrow v_0^{\ 2}=64(550)$, and $v_0=8\sqrt{550}=40\sqrt{22}\approx 187.62$ feet per second.

Lesson Nineteen

1. Find the sum: $\sum_{i=1}^{6}(3i+2)$.

$$\sum_{i=1}^{6}(3i+2)=(5+8+11+14+17+20)=75.$$

2. Find the sum: $\sum_{k=0}^{4}\frac{1}{k^2+1}$.

$$\sum_{k=0}^{4}\frac{1}{k^2+1}=\frac{1}{0+1}+\frac{1}{1+1}+\frac{1}{4+1}+\frac{1}{9+1}+\frac{1}{16+1}=1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}=\frac{158}{85}.$$

3. Use sigma notation to write the sum $\frac{1}{5(1)}+\frac{1}{5(2)}+\frac{1}{5(3)}+\cdots+\frac{1}{5(11)}$.

$$\frac{1}{5(1)}+\frac{1}{5(2)}+\frac{1}{5(3)}+\cdots+\frac{1}{5(11)}=\sum_{i=1}^{11}\frac{1}{5i}.$$

4. Use sigma notation to write the sum $\left[2\left(1+\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)+\cdots+\left[2\left(1+\frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$.

$$\left[2\left(1+\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)+\cdots+\left[2\left(1+\frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)=\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1+\frac{3i}{n}\right)^2\right].$$

5. Use the formulas for summations to evaluate $\sum_{i=1}^{24}4i$.

$$\sum_{i=1}^{24}4i=4\sum_{i=1}^{24}i=4\left[\frac{24(24+1)}{2}\right]=4\left[\frac{24(25)}{2}\right]=1200.$$

6. Use the formulas for summations to evaluate $\sum_{i=1}^{20}(i-1)^2$.

$$\sum_{i=1}^{20}(i-1)^2=\sum_{i=1}^{19}i^2=\left[\frac{19(19+1)(2(19)+1)}{6}\right]=\left[\frac{19(20)(39)}{6}\right]=2470.$$

7. Use $n=4$ rectangles and right endpoints to approximate the area of the region bounded by $f(x)=2x+5,\ 0\le x\le 2,$ in the first quadrant.

$\Delta x=\frac{2-0}{4}=\frac{1}{2}$. Using right endpoints, you obtain

$$A\approx\frac{1}{2}\left[f(0.5)+f(1)+f(1.5)+f(2)\right]=\frac{1}{2}\left[6+7+8+9\right]=15.$$

8. Use $n = 4$ rectangles and right endpoints to approximate the area of the region bounded by $f(x) = \cos x,\ 0 \le x \le \frac{\pi}{2}$, in the first quadrant.

$\Delta x = \dfrac{\frac{\pi}{2} - 0}{4} = \dfrac{\pi}{8}$. Using right endpoints, you obtain

$$A \approx \frac{\pi}{8}\left[\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right)\cos\left(3\frac{\pi}{8}\right)\cos\left(\frac{\pi}{2}\right)\right] \approx 0.7908.$$

9. Use the limit definition to find the area of the region between $f(x) = -4x + 5,\ 0 \le x \le 1$, and the x-axis.

We see that $\Delta x = \dfrac{1}{n}$. Using right endpoints, you have

$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-4\left(\frac{i}{n}\right) + 5\right]\left(\frac{1}{n}\right) = \frac{-4}{n^2}\left(\sum_{i=1}^{n} i\right) + 5 = \left(\frac{-4}{n^2}\right)\frac{n(n+1)}{2} + 5 = -2\left(1 + \frac{1}{n}\right) + 5.$$

As n tends to infinity, $s(n)$ tends to $-2 + 5 = 3$. The area is 3.

10. Use the limit definition to find the area of the region between $f(x) = x^2 + 1,\ 0 \le x \le 3,$ and the x-axis.

We see that $\Delta x = \dfrac{3}{n}$. Using right endpoints, you have

$$S(n) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right) = \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1 = \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n)$$
$$= \frac{9}{2}\frac{2n^2 + 3n + 1}{n^2} + 3.$$

As n tends to infinity, $S(n)$ tends to $\dfrac{9}{2}(2) + 3 = 12$. The area is 12.

11. Use a geometric formula to determine the definite integral: $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$.

This definite integral represents the area of a semicircle of radius 3. Hence, the value of the integral is $\dfrac{1}{2}\pi(3)^2 = \dfrac{9\pi}{2}$.

Lesson Twenty

1. Evaluate the definite integral: $\int_0^2 6x\,dx$.

$$\int_0^2 6x\,dx = \left[3x^2\right]_0^2 = 3(2)^2 - 3(0)^2 = 12.$$

2. Evaluate the definite integral: $\int_0^1 (2t-1)^2\,dt$.

$$\int_0^1 (2t-1)^2\,dt = \int_0^1 \left(4t^2 - 4t + 1\right)dt = \left[\frac{4t^3}{3} - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}.$$

3. Evaluate the definite integral: $\int_1^4 \frac{u-2}{\sqrt{u}}\,du$.

$$\int_1^4 \frac{u-2}{\sqrt{u}}\,du = \int_1^4 \left(u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\right)du = \left[\frac{2}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right]_1^4 = \left(\frac{2}{3}(8) - 4(2)\right) - \left(\frac{2}{3}(1) - 4\right) = \frac{14}{3} - 4 = \frac{2}{3}.$$

4. Evaluate the definite integral: $\int_0^5 |2x-5|\,dx$.

We have to split up the integral at the point $x = \frac{5}{2}$ to remove the absolute value sign.

$$\begin{aligned}
\int_0^5 |2x-5|\,dx &= \int_0^{5/2} (5-2x)\,dx + \int_{5/2}^5 (2x-5)\,dx \\
&= \left[5x - x^2\right]_0^{5/2} + \left[x^2 - 5x\right]_{5/2}^5 \\
&= \left(\frac{25}{2} - \frac{25}{4}\right) + \left((25-25) - \left(\frac{25}{4} - \frac{25}{2}\right)\right) \\
&= \frac{25}{4} + \frac{25}{4} \\
&= \frac{25}{2}.
\end{aligned}$$

5. Evaluate the definite integral of the trigonometric function: $\int_0^\pi (1 + \sin x)\,dx$.

$$\int_0^\pi (1 + \sin x)\,dx = \left[x - \cos x\right]_0^\pi = \left(\pi - (-1)\right) - \left(0 - 1\right) = 2 + \pi.$$

6. Evaluate the definite integral of the trigonometric function: $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,dx$.

$$\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,dx = \left[4\sec\theta\right]_{-\pi/3}^{\pi/3} = 4\left(2-2\right) = 0.$$

7. Find the area of the region bounded by the graphs of the equations $y = 5x^2 + 2$, $x = 0$, $x = 2$, and $y = 0$.

$$A = \int_0^2 \left(5x^2 + 2\right)dx = \left[\frac{5x^3}{3} + 2x\right]_0^2 = \frac{40}{3} + 4 = \frac{52}{3}.$$

8. Find the area of the region bounded by the graphs of the equations $y = x^3 + x,\ x = 2,$ and $y = 0$.

$$A = \int_0^2 \left(x^3 + x\right) dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2 = \frac{16}{4} + \frac{4}{2} = 6.$$

9. Find the value(s) of c guaranteed by the mean value theorem for integrals for the function $f(x) = x^3,\ 0 \le x \le 3$.

$\int_a^b f(x)\,dx = f(c)(b-a) \Rightarrow \int_0^3 x^3\,dx = c^3(3-0)$. Now we have $\int_0^3 x^3\,dx - \left[\frac{x^4}{4}\right]_0^3 = \frac{81}{4}$, which gives

$$\frac{81}{4} = c^3(3) \Rightarrow c^3 = \frac{27}{4} \Rightarrow \frac{3}{4^{1/3}} \approx 1.8899.$$

10. Find the value(s) of c guaranteed by the mean value theorem for integrals for the function $f(x) = \cos x,\ -\frac{\pi}{3} \le x \le \frac{\pi}{3}$.

$\int_a^b f(x)\,dx = f(c)(b-a) \Rightarrow \int_{-\pi/3}^{\pi/3} \cos x\,dx = \cos(c)\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right) = \cos(c)\frac{2\pi}{3}$. Now we have

$\int_{-\pi/3}^{\pi/3} \cos x\,dx = \left[\sin x\right]_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$, which gives $\sqrt{3} = \cos(c)\frac{2\pi}{3} \Rightarrow \cos(c) = \frac{3\sqrt{3}}{2\pi}$.

Using a graphing utility, you obtain $c \approx \pm 0.5971$.

11. Find the average value of the function $f(x) = 4x^3 - 3x^2$ on the interval $\left[-1, 2\right]$.

$\int_{-1}^2 \left(4x^3 - 3x^2\right) dx = \left[x^4 - x^3\right]_{-1}^2 = (16-8) - (1+1) = 6$. The average value is

$$\frac{1}{2-(-1)} \int_{-1}^2 \left(4x^3 - 3x^2\right) dx = \frac{6}{3} = 2.$$

Lesson Twenty-One

1. Let $F(x) = \int_0^x (4t-7)\,dt$, and evaluate F at $x=2$, $x=5$, and $x=8$.

$F(x) = \int_0^x (4t-7)\,dt = \left[2t^2-7t\right]_0^x = 2x^2-7x$. We can now evaluate F at the 3 points:

$F(2)=2(2)^2-7(2)=-6$; $F(5)=2(5)^2-7(5)=15$; and $F(8)=2(8)^2-7(8)=72$.

2. Let $F(x) = \int_2^x \frac{-2}{t^3}\,dt$, and evaluate F at $x=2$, $x=5$, and $x=8$.

$F(x) = \int_2^x \frac{-2}{t^3}\,dt = -\int_2^x 2t^{-3}\,dt = \left[\frac{1}{t^2}\right]_2^x = \frac{1}{x^2}-\frac{1}{4}$. We can now evaluate F at the 3 points:

$F(2)=\frac{1}{4}-\frac{1}{4}=0$; $F(5)=\frac{1}{25}-\frac{1}{4}=-\frac{21}{100}$; and $F(8)=\frac{1}{64}-\frac{1}{4}=-\frac{15}{64}$.

3. **a.** Integrate the function $F(x)=\int_0^x (t+2)\,dt$ to find F as a function of x.

b. Illustrate the second fundamental theorem of calculus by differentiating the result from part a.

a. $F(x) = \int_0^x (t+2)\,dt = \left[\frac{t^2}{2}+2t\right]_0^x = \frac{x^2}{2}+2x.$

b. $\frac{d}{dx}\left[\frac{x^2}{2}+2x\right]=x+2$ (the original integrand).

4. **a.** Integrate the function $F(x)=\int_{\pi/4}^x \sec^2 t\,dt$ to find F as a function of x.

b. Illustrate the second fundamental theorem of calculus by differentiating the result from part a.

a. $F(x) = \int_{\pi/4}^x \sec^2 t\,dt = \left[\tan t\right]_{\pi/4}^x = \tan x - 1.$

b. $\frac{d}{dx}\left[\tan x - 1\right]=\sec^2 x$ (the original integrand).

5. Use the second fundamental theorem of calculus to find $F'(x)$ if $F(x)=\int_0^x t\cos t\,dt$.

By the second fundamental theorem of calculus, the derivative of this integral is the original integrand, in the variable x: $F'(x) = \frac{d}{dx}\int_0^x t\cos t\,dt = x\cos x.$

6. Use the second fundamental theorem of calculus to find $F'(x)$ if $F(x)=\int_1^x \sqrt[4]{t}\,dt$.

By the second fundamental theorem of calculus, the derivative of this integral is the original integrand, in the variable x: $F'(x) = \frac{d}{dx}\int_1^x \sqrt[4]{t}\,dt = \sqrt[4]{x}.$

7. Find $F'(x)$ if $F(x)=\int_x^2 t^3\,dt$.

$$F'(x) \;=\; \frac{d}{dx}\int_x^2 t^3\,dt \;=\; -\frac{d}{dx}\int_2^x t^3\,dt \;=\; -x^3.$$

You could also have first integrated and then differentiated the function.

8. Find $F'(x)$ if $F(x)=\int_0^{\sin x}\sqrt{t}\,dt$.

We need to use the chain rule. $F'(x) \;=\; \frac{d}{dx}\int_0^{\sin x}\sqrt{t}\,dt \;=\; \sqrt{\sin x}\,\frac{d}{dx}[\sin x] \;=\; \sqrt{\sin x}(\cos x).$

You could also have first integrated and then differentiated the function.

9. Find $F'(x)$ if $F(x)=\int_x^{x+2}(4t+1)\,dt$.

$$F(x) \;=\; \int_x^{x+2}(4t+1)\,dt \;=\; \int_x^0(4t+1)\,dt+\int_0^{x+2}(4t+1)\,dt \;=\; -\int_0^x(4t+1)\,dt+\int_0^{x+2}(4t+1)\,dt.$$

$F'(x)=-(4x+1)+4(x+2)+1=8.$

10. A particle is moving along a line with velocity function $v(t)=t^2-t-12$ feet per second. Find the displacement on the interval $1\le t\le 5$ and the total distance traveled on that interval.

We set up the integrals and leave it to you to do the evaluations.

Displacement $=\int_1^5 v(t)\,dt \;=\; \int_1^5(t^2-t-12)\,dt \;=\; -\frac{56}{3}$. (This indicates $\frac{56}{3}$ feet to the left.)

For the total distance traveled, we notice that $v(t)=t^2-t-12=(t-4)(t+3)$, so the particle changes direction at 4.

Total distance traveled: $\int_1^5|v(t)|\,dt \;=\; \int_1^5\left|t^2-t-12\right|dt \;=\; \int_1^4-\left(t^2-t-12\right)dt \;+\; \int_4^5\left(t^2-t-12\right)dt \;=$

$\frac{45}{2}+\frac{23}{6}=\frac{79}{3}$.

Lesson Twenty-Two

1. Find the integral: $\int (1+6x)^4 (6)\,dx$.

Let $u = 1+6x,\ du = 6\,dx$. Then $\int (1+6x)^4 (6)\,dx = \dfrac{(1+6x)^5}{5} + C.$

2. Find the integral: $\int \sqrt{25-x^2}\,(-2x)\,dx$.

Let $u = 25-x^2,\ du = -2x\,dx$. Then $\int \sqrt{25-x^2}\,(-2x)\,dx \;=\; \int (25-x^2)^{\frac{1}{2}}(-2x)\,dx$

$$= \frac{(25-x^2)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}(25-x^2)^{\frac{3}{2}} + C.$$

3. Find the integral: $\int t\sqrt{t^2+2}\,dt$.

Let $u = t^2+2,\ du = 2t\,dt$. Then $\int t\sqrt{t^2+2}\,dt \;=\; \dfrac{1}{2}\int (t^2+2)^{\frac{1}{2}}(2t)\,dt \;=\; \dfrac{1}{2}\dfrac{(t^2+2)^{\frac{3}{2}}}{\frac{3}{2}} + C \;=\; \dfrac{1}{3}(t^2+2)^{\frac{3}{2}} + C.$

4. Find the integral: $\int \dfrac{x}{(1-x^2)^3}\,dx$.

Let $u = 1-x^2,\ du = -2x\,dx$.

$$\int \frac{x}{(1-x^2)^3}\,dx \;=\; \frac{-1}{2}\int (1-x^2)^{-3}(-2x)\,dx \;=\; \frac{-1}{2}\frac{(1-x^2)^{-2}}{-2} + C \;=\; \frac{1}{4(1-x^2)^2} + C.$$

5. Find the integral: $\int \sin 4x\,dx$.

Let $u = \sin 4x,\ du = 4\,dx$. Then $\int \sin 4x\,dx \;=\; \dfrac{1}{4}\int \sin(4x)(4)\,dx \;=\; \dfrac{1}{4}(-\cos 4x) + C \;=\; -\dfrac{1}{4}\cos 4x + C.$

6. Find the integral: $\int \tan^4 x \sec^2 x\,dx$.

Let $u = \tan x,\ du = \sec^2 x\,dx$. Then $\int \tan^4 x \sec^2 x\,dx \;=\; \dfrac{\tan^5 x}{5} + C.$

7. Evaluate the definite integral: $\int_0^4 \dfrac{1}{\sqrt{2x+1}}\,dx$.

Let $u = 2x+1,\ du = 2\,dx$. Then

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx \;=\; \frac{1}{2}\int_0^4 (2x+1)^{-\frac{1}{2}}(2)\,dx \;=\; \left[\frac{1}{2}\cdot\frac{(2x+1)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^4 \;=\; \left[\sqrt{2x+1}\right]_0^4 \;=\; \sqrt{9}-\sqrt{1} \;=\; 2.$$

8. Evaluate the definite integral: $\int_0^{\pi/2} \cos\left(\dfrac{2x}{3}\right)dx$.

Let $u = \dfrac{2x}{3},\ du = \dfrac{2}{3}\,dx$. Then

$$\int_0^{\pi/2} \cos\left(\frac{2x}{3}\right)dx \;=\; \frac{3}{2}\int_0^{\pi/2} \cos\left(\frac{2x}{3}\right)\frac{2}{3}\,dx \;=\; \frac{3}{2}\left[\sin\left(\frac{2x}{3}\right)\right]_0^{\pi/2} \;=\; \frac{3}{2}\sin\frac{\pi}{3} \;=\; \frac{3\sqrt{3}}{4}.$$

9. Evaluate the integral using the properties of even and odd functions: $\int_{-2}^{2} x^2\left(x^2+1\right)dx$.

The function is even, so we have

$$\int_{-2}^{2} x^2\left(x^2+1\right)dx = 2\int_{0}^{2} x^2\left(x^2+1\right)dx = 2\int_{0}^{2}\left(x^4+x^2\right)dx = 2\left[\frac{x^5}{5}+\frac{x^3}{3}\right]_0^2 = 2\left(\frac{32}{5}+\frac{8}{3}\right) = \frac{272}{15}.$$

10. Evaluate the integral using the properties of even and odd functions: $\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx$.

The function is odd, and the interval is symmetric about the origin, so the integral is 0.

Lesson Twenty-Three

1. Use the trapezoidal rule and $n=4$ to approximate $\int_0^2 x^3 dx$.

$$n=4,\ b-a=2-0=2,\ \frac{(b-a)}{2n}=\frac{1}{4}.$$

$$\int_0^2 x^3 dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]$$

$$=\frac{1}{4}\left[0+2\left(\frac{1}{2}\right)^3+2(1)^3+2\left(\frac{3}{2}\right)^3+(2)^3\right]=\frac{17}{4}=4.25.$$

2. Use the trapezoidal rule and $n=4$ to approximate $\int_0^2 x\sqrt{x^2+1}\,dx$.

$$n=4,\ b-a=2-0=2,\ \frac{(b-a)}{2n}=\frac{1}{4}.$$

$$\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]$$

$$=\frac{1}{4}\left[0+2\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1}+2(1)\sqrt{(1)^2+1}\right]+2\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1}+2\sqrt{(2)^2+1}$$

$$\approx 3.457.$$

3. Use the trapezoidal rule and $n=8$ to approximate $\int_0^8 \sqrt[3]{x}\,dx$.

$$n=8,\ b-a=8-0=8,\ \frac{(b-a)}{2n}=\frac{1}{2}.$$

$$\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}\left[0+2+2\sqrt[3]{2}+2\sqrt[3]{3}+2\sqrt[3]{4}+2\sqrt[3]{5}+2\sqrt[3]{6}+2\sqrt[3]{7}+2\right] \approx 11.7296.$$

4. Approximate the integral $\int_0^2 \sqrt{1+x^3}\,dx$ with the trapezoidal rule and $n=4$. Compare your answer with the approximation using a graphing utility.
The trapezoidal approximation is 3.283; a graphing utility gives 3.241.

5. Approximate the integral $\int_3^{3.1} \cos x^2\,dx$ with the trapezoidal rule and $n=4$. Compare your answer with the approximation using a graphing utility.
The trapezoidal approximation is -0.098; a graphing utility gives -0.098.

6. The elliptic integral $8\sqrt{3}\int_0^{\pi/2}\sqrt{1-\frac{2}{3}\sin^2\theta}\,d\theta$ gives the circumference of an ellipse. Approximate this integral with a numerical technique.
Your approximation should be close to 17.4755.

7. Later in this course, you will study inverse trigonometric functions. You will learn that π can be expressed as a definite integral: $\pi = \int_0^1 \frac{4}{1+x^2}dx$. Use a numerical technique to approximate this integral, and compare your answer to π.
Your approximation should be close to 3.14159.

8. Suppose f is concave downward on the interval $[0,2]$. Using the trapezoidal rule, would your answer be too large or too small compared to the exact answer?

Your answer would be too small because the trapezoids are under the curve.

Lesson Twenty-Four

1. Use the properties of logarithms to expand the expression $\ln\frac{x}{4}$.

$\ln\frac{x}{4} = \ln x - \ln 4.$

2. Use the properties of logarithms to expand the expression $\ln\left(x\sqrt{x^2+5}\right)$.

$\ln\left(x\sqrt{x^2+5}\right) = \ln x + \ln\sqrt{x^2+5} = \ln x + \ln(x^2+5)^{\frac{1}{2}} = \ln x + \frac{1}{2}\ln(x^2+5).$

3. Write the expression $3\ln x + 2\ln y - 4\ln z$ as a logarithm of a single quantity.

$3\ln x + 2\ln y - 4\ln z = \ln x^3 + \ln y^2 - \ln z^4 = \ln\left(\frac{x^3y^2}{z^4}\right).$

4. Write the expression $2\ln 3 - \frac{1}{2}\ln(x^2+1)$ as a logarithm of a single quantity.

$2\ln 3 - \frac{1}{2}\ln(x^2+1) = \ln 3^2 - \ln\sqrt{x^2+1} = \ln\frac{9}{\sqrt{x^2+1}}.$

5. Find the derivative of the function $f(x) = \ln(3x)$.

$f'(x) = \frac{1}{3x}(3) = \frac{1}{x}$.

6. Find the derivative of the function $f(x) = (\ln x)^4$.

$f'(x) = 4(\ln x)^3\frac{d}{dx}[\ln x] = 4(\ln x)^3\frac{1}{x} = \frac{4(\ln x)^3}{x}$.

7. Find the derivative of the function $f(x) = x^2\ln x$.

We use the product rule: $f'(x) = x^2\left(\frac{1}{x}\right) + (\ln x)(2x) = x + 2x\ln x.$

8. Find the derivative of the function $f(x) = \ln(\ln x^2)$.

We use the chain rule:

$f'(x) = \frac{1}{\ln x^2}\frac{d}{dx}(\ln x^2) = \frac{1}{2\ln x}\frac{1}{x^2}(2x) = \frac{1}{x\ln x}.$

9. Find the derivative of the function $f(x) = \ln|\sec x + \tan x|$. Simplify your answer.

We use the chain rule:

$$\begin{aligned}
f'(x) &= \frac{1}{\sec x + \tan x}\frac{d}{dx}[\sec x + \tan x] \\
&= \frac{1}{\sec x + \tan x}\left(\sec x\tan x + \sec^2 x\right) \\
&= \frac{1}{\sec x + \tan x}(\sec x)(\tan x + \sec x) \\
&= \sec x.
\end{aligned}$$

10. Find an equation of the tangent line to the graph of $y = \ln x^3$ at the point $(1,0)$.

$y = \ln x^3 = 3\ln x \Rightarrow y' = \dfrac{3}{x}$. The slope at $x = 1$ is 3, and the equation of the tangent line is $y - 0 = 3(x-1)$, or $y = 3x - 3$.

11. Use implicit differentiation to find y' if $\ln xy + 5x = 30$.

We assume here that both x and y are positive.

$$\frac{1}{xy}\left[xy' + y\right] + 5 = 0 \Rightarrow xy' + y + 5xy = 0 \Rightarrow xy' = -y - 5xy.$$

$$y' = \frac{-y - 5xy}{x}.$$

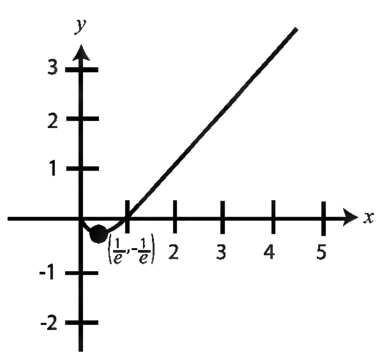
12. Use logarithmic differentiation to find y' if $y = \sqrt{\dfrac{x^2-1}{x^2+1}}$, $x > 1$.

$$\ln y = \ln\sqrt{\frac{x^2-1}{x^2+1}} = \frac{1}{2}\ln\left[\frac{x^2-1}{x^2+1}\right] = \frac{1}{2}\left[\ln(x^2-1) - \ln(x^2+1)\right].$$

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right] \Rightarrow y' = \frac{1}{2}y\left[\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right] = y = \frac{1}{2}\sqrt{\frac{x^2-1}{x^2+1}}\left[\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right].$$

13. Use your knowledge of graphing to graph the function $y = x\ln x$.

The domain is $x > 0$. The derivatives are as follows: $y' = 1 + \ln x$, and $y'' = \dfrac{1}{x}$. Hence, there is a critical number at $x = \dfrac{1}{e}$. The graph is concave upward on its domain, so there is a relative minimum at $\left(\dfrac{1}{e}, -\dfrac{1}{e}\right)$.



Lesson Twenty-Five

1. Find the indefinite integral: $\int \frac{1}{2x+5}\,dx$.

Let $u = 2x+5,\ du = 2\,dx$.

$$\int \frac{1}{2x+5}\,dx \;=\; \frac{1}{2}\int \frac{1}{2x+5}(2)\,dx \;=\; \frac{1}{2}\ln|2x+5| + C.$$

2. Find the indefinite integral: $\int \frac{x^2-4}{x}\,dx$.

$$\int \frac{x^2-4}{x}\,dx \;=\; \int\left(x - \frac{4}{x}\right)dx \;=\; \frac{x^2}{2} - 4\ln|x| + C.$$

3. Find the indefinite integral: $\int \frac{x^2-3x+2}{x+1}\,dx$.

First do long division to show that $\frac{x^2-3x+2}{x+1} = x - 4 + \frac{6}{x+1}$. Then

$$\int \frac{x^2-3x+2}{x+1}\,dx \;=\; \int\left(x - 4 + \frac{6}{x+1}\right)dx \;=\; \frac{x^2}{2} - 4x + 6\ln|x+1| + C.$$

4. Find the indefinite integral: $\int \tan 5\theta\, d\theta$.

$$\int \tan 5\theta\, d\theta \;=\; \frac{1}{5}\int (\tan 5\theta)(5)\,d\theta \;=\; -\frac{1}{5}\ln|\cos 5\theta| + C.$$

5. Find the indefinite integral: $\int \frac{\sec x \tan x}{\sec x - 1}\,dx$.

Let $u = \sec x - 1,\ du = \sec x \tan x\,dx$. Then $\int \frac{\sec x \tan x}{\sec x - 1}\,dx \;=\; \ln|\sec x - 1| + C.$

6. Solve the differential equation $\frac{dy}{dx} = \frac{3}{2-x}$.

$$y \;=\; \int \frac{3}{2-x}\,dx \;=\; -3\int \frac{1}{x-2}\,dx \;=\; -3\ln|x-2| + C.$$

7. Evaluate the definite integral: $\int_1^e \frac{(1+\ln x)^2}{x}\,dx$.

Let $u = 1 + \ln x,\ du = \frac{1}{x}\,dx$.

$$\int_1^e \frac{(1+\ln x)^2}{x}\,dx \;=\; \left[\frac{(1+\ln x)^3}{3}\right]_1^e \;=\; \frac{(1+1)^3}{3} - \frac{(1+0)^3}{3} \;=\; \frac{7}{3}.$$

8. Evaluate the definite integral: $\int_0^1 \frac{x-1}{x+1}\,dx$.

Use long division to write the integrand as $\frac{x-1}{x+1} = 1 - \frac{2}{x+1}$.

$$\int_0^1 \frac{x-1}{x+1}\,dx \;=\; \int_0^1 \left(1 - \frac{2}{x+1}\right)dx \;=\; \Big[x - 2\ln|x+1|\Big]_0^1 \;=\; (1 - 2\ln 2) - 0 \;=\; 1 - \ln 4.$$

9. Find the area of the region bounded by the equations $y = \frac{x^2+4}{x}$, $x=1$, $x=4$, and $y=0$.

$$\text{Area} = \int_1^4 \frac{x^2+4}{x}\,dx \;=\; \int_1^4 \left(x + \frac{4}{x}\right)dx \;=\; \left[\frac{x^2}{2} + 4\ln|x|\right]_1^4 \;=\; (8 + 4\ln 4) - \left(\frac{1}{2} + 0\right) \;=\; \frac{15}{2} + \ln 256.$$

10. Find the average value of the function $f(x) = \frac{8}{x^2}$ over the interval $[2,4]$.

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx \;=\; \frac{1}{4-2}\int_2^4 8x^{-2}\,dx \;=\; 4\left[\frac{-1}{x}\right]_2^4 \;=\; 4\left(\frac{-1}{4} + \frac{1}{2}\right) \;=\; 1.$$

11. Show that $f(x) = \sqrt{x-4}$ and $g(x) = x^2 + 4,\ x \ge 0,$ are inverse functions.

Notice that the domain of f is $[4,\infty)$ and the domain of g is $[0,\infty)$.

$f(g(x)) = f(x^2+4) = \sqrt{x^2+4-4} = \sqrt{x^2} = x$ (since $x > 0$). Similarly, you can show that $g(f(x)) = x$.

12. Does the function $f(x) = \tan x$ have an inverse function? Why or why not?

The function is not one-to-one, so it does not have an inverse. However, if you restrict the domain to the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then it has an inverse, called the arctangent function.

Lesson Twenty-Six

1. Solve for x if $e^x = 12$.

Take the logarithm of both sides of the equation: $\ln(e^x) = \ln 12 \Rightarrow x = \ln 12 \approx 2.485$.

2. Solve for x if $\ln x = 2$.

Raise both sides to the power e: $e^{\ln x} = e^2 \Rightarrow x = e^2 \approx 7.389$.

3. Solve for x if $\ln\sqrt{x+2} = 1$.

$\ln\sqrt{x+2} = 1 \Rightarrow e^{\ln\sqrt{x+2}} = \sqrt{x+2} = e^1 \Rightarrow x+2 = e^2 \Rightarrow x = e^2 - 2 \approx 5.389$.

4. Find the derivative of the function $f(x) = e^{\sqrt{x}}$.

By the chain rule, $f'(x) = e^{\sqrt{x}}\left(\frac{d}{dx}\left[x^{\frac{1}{2}}\right]\right) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$.

5. Find the derivative of the function $f(x) = \frac{e^x+1}{e^x-1}$.

Use the quotient rule:

$$f'(x) = \frac{(e^x-1)e^x-(e^x+1)e^x}{(e^x-1)^2} = \frac{-2e^x}{(e^x-1)^2}.$$

6. Find the derivative of the function $f(x) = e^x(\sin x + \cos x)$.

Use the product rule:

$f'(x) = e^x(\cos x - \sin x) + (\sin x + \cos x)e^x = 2e^x\cos x$.

7. Find the equation of the tangent line to the graph of $y = xe^x - e^x$ at the point $(1,0)$.

$y' = xe^x + e^x - e^x = xe^x$, so the slope at the point $x = 1$ is e. The equation of the tangent line at the given point is $y - 0 = e(x-1)$, or $y = ex - e$.

8. Find the indefinite integral: $\int x^2e^{x^3}\,dx$.

Let $u = x^3$, $du = 3x^2\,dx$.

$$\int x^2e^{x^3}\,dx = \frac{1}{3}\int e^{x^3}3x^2dx = \frac{1}{3}e^{x^3}+C.$$

9. Find the indefinite integral: $\int\frac{e^{-x}}{1+e^{-x}}dx$.

Let $u = 1+e^{-x}$, $du = -e^{-x}\,dx$.

$$\int\frac{e^{-x}}{1+e^{-x}}dx = -\int\frac{1}{1+e^{-x}}\left(-e^{-x}\right)dx = -\ln\left(1+e^{-x}\right)+C.$$

10. Evaluate the definite integral: $\int_3^4 e^{3-x}\,dx$.

Let $u = 3-x$, $du = -dx$.

$$\int_3^4 e^{3-x}\,dx = -\int_3^4 e^{3-x}(-dx) = -\left[e^{3-x}\right]_3^4 = -(e^{-1}-1) = 1-\frac{1}{e}.$$

11. Evaluate the definite integral: $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$.

Let $u = \frac{3}{x}$, $du = \frac{-3}{x^2}dx$.

$$\int_1^3 \frac{e^{3/x}}{x^2}\,dx \;=\; -\frac{1}{3}\int_1^3 e^{\frac{3}{x}}\left(\frac{-3}{x^2}\right)dx \;=\; -\frac{1}{3}\left[e^{\frac{3}{x}}\right]_1^3 \;=\; -\frac{1}{3}\left(e-e^3\right) \;=\; \frac{e^3-e}{3}.$$

12. Use implicit differentiation to find $\frac{dy}{dx}$ if $xe^y - 10x + 3y = 0$.

$$\begin{aligned} xe^y y' + e^y - 10 + 3y' &= 0 \\ y'(xe^y + 3) &= 10 - e^y \\ y' &= \frac{10-e^y}{xe^y+3}. \end{aligned}$$

13. Find the second derivative of the function $f(x) = (3+2x)e^{-3x}$.

$$f'(x) = (3+2x)e^{-3x}(-3) + 2e^{-3x} = e^{-3x}(-6x-7).$$
$$f''(x) = e^{-3x}(-6) + (-6x-7)e^{-3x}(-3) = e^{-3x}(18x+15).$$

Lesson Twenty-Seven

1. Solve for x if $\log_3 x = -1$.

Apply the exponential function to base 3 to both sides:

$$3^{\log_3 x} = 3^{-1} \Rightarrow x = \frac{1}{3}.$$

2. Solve for x if $x^2 - x = \log_5 25$.

$x^2 - x = \log_5 25 = \log_5 5^2 = 2.$

$x^2 - x - 2 = (x-2)(x+1) = 0.$

Hence, $x = -1,\ 2$. You can check that both solutions are valid.

3. Solve for x if $3^{2x} = 75$.

$$\log_3\left(3^{2x}\right) = \log_3 75 \Rightarrow 2x = \log_3 75 \Rightarrow x = \frac{\log_3 75}{2} \approx 1.965.$$

4. Find the derivative of the function $f(x) = 5^{-4x}$.

$$f'(x) = 5^{-4x} \ln 5 \frac{d}{dx}\left[-4x\right] = -4(5^{-4x}) \ln 5.$$

5. Find the derivative of the function $f(x) = x9^x$.

Use the product rule: $f'(x) = x(9^x \ln 9) + 9^x = 9^x(1 + x\ln 9).$

6. Find the derivative of the function $f(x) = \log_5 \sqrt{x^2 - 1}$.

$$f(x) = \log_5 \sqrt{x^2 - 1} = \frac{1}{2}\log_5(x^2 - 1).$$

$$f'(x) = \frac{1}{2}\left(\frac{1}{\ln 5}\right)\frac{1}{(x^2-1)}(2x) = \frac{x}{(x^2-1)\ln 5}.$$

7. Find the equation of the tangent line to the graph of $y = \log_{10} 2x$ at the point $(5,1)$.

$y' = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{2x}\right)(2) = \frac{1}{x\ln 10}$. The slope at the point $x = 5$ is $\frac{1}{5\ln 10}$, and the equation of the tangent line is $y - 1 = \frac{1}{5\ln 10}(x-5)$, or $y = \frac{1}{5\ln 10}x + 1 - \frac{1}{\ln 10}$.

8. Find the indefinite integral: $\int 5^{-x}\,dx$.

$$\int 5^{-x}\,dx = -\int 5^{-x}(-dx) = \frac{-5^{-x}}{\ln 5} + C.$$

9. Find the indefinite integral: $\int\left(x^3 + 3^{-x}\right)dx$.

$$\int\left(x^3 + 3^{-x}\right)dx = \frac{x^4}{4} - \frac{3^{-x}}{\ln 3} + C.$$

10. Evaluate the definite integral: $\int_1^e \left(6^x - 2^x\right)dx$.

$$\int_1^e \left(6^x - 2^x\right) dx = \left[\frac{6^x}{\ln 6} - \frac{2^x}{\ln 2}\right]_1^e = \left(\frac{6^e}{\ln 6} - \frac{2^e}{\ln 2}\right) - \left(\frac{6^1}{\ln 6} - \frac{2^1}{\ln 2}\right) = \frac{6^e - 6}{\ln 6} - \frac{2^e - 2}{\ln 2}.$$

11. You deposit \$1000 in an account that pays 5% interest for 30 years. How much will be in the account if the interest is compounded

a. monthly?

b. continuously?

a. $A = P\left(1 + \frac{r}{n}\right)^{nt} = 1000\left(1 + \frac{0.05}{12}\right)^{(12)(30)} \approx \$4467.74.$

b. $A = Pe^{rt} = 1000e^{(0.05)(30)} \approx \$4481.69.$

12. Use logarithmic differentiation to find $\frac{dy}{dx}$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $y = x^{\sin x}$.

$\ln y = \ln x^{\sin x} = (\sin x)(\ln x).$

$$\frac{y'}{y} = (\sin x)\frac{1}{x} + (\ln x)\cos x$$

$$y' = y\left[(\sin x)\frac{1}{x} + (\ln x)\cos x\right].$$

At the given point, $y' = \frac{\pi}{2}\left[1\frac{2}{\pi} + 0\right] = 1$.

Lesson Twenty-Eight

1. Evaluate the expression $\arcsin 0$ without a calculator.

$\arcsin 0 = 0$ because $\sin 0 = 0$ and $\frac{-\pi}{2} \le 0 \le \frac{\pi}{2}$.

2. Evaluate the expression $\arccos 1$ without a calculator.

$\arccos 1 = 0$ because $\cos 0 = 1$ and $0 \le 0 \le \pi$.

3. Use a right triangle to evaluate the expression $\sin\left(\arctan\frac{3}{4}\right)$.

Draw a right triangle with acute angle y, opposite side 3, and adjacent side 4. By the Pythagorean theorem, the hypotenuse is 5. From the triangle, we see that $\sin\left(\arctan\frac{3}{4}\right) = \sin y = \frac{3}{5}$.

4. Use a right triangle to evaluate the expression $\sec\left(\arcsin\frac{4}{5}\right)$.

Draw a right triangle with acute angle y, opposite side 4, and hypotenuse 5. By the Pythagorean theorem, the adjacent side is 3. From the triangle, we see that $\sec\left(\arcsin\frac{4}{5}\right) = \sec y = \frac{5}{3}$.

5. Find the derivative of the function $f(x) = 2\arcsin(x-1)$.

$$f'(x) = 2\frac{1}{\sqrt{1-(x-1)^2}} = \frac{2}{\sqrt{2x-x^2}}[0,1].$$

6. Find the derivative of the function $f(x) = \arctan e^x$.

$$f'(x) = \frac{1}{1+(e^x)^2}\left(e^x\right) = \frac{e^x}{1+e^{2x}}.$$

7. Find the derivative of the function $f(x) = \operatorname{arcsec} 2x$.

$$f'(x) = \frac{1}{|2x|\sqrt{(2x)^2-1}}(2) = \frac{1}{|x|\sqrt{4x^2-1}}.$$

8. Find the equation of the tangent line to the graph of $y = \arctan\frac{x}{2}$ at the point $\left(2, \frac{\pi}{4}\right)$.

$y' = \frac{1}{1+\left(\frac{x}{2}\right)^2}\left(\frac{1}{2}\right) = \frac{2}{4+x^2}$. At the point $x = 2$, the slope is $\frac{1}{4}$. The equation of the tangent line is

$y - \frac{\pi}{4} = \frac{1}{4}(x-2)$, or $y = \frac{1}{4}x + \frac{\pi}{4} - \frac{1}{2}$.

9. Find the indefinite integral: $\int \frac{1}{\sqrt{9-x^2}}dx$.

$$\int \frac{1}{\sqrt{9-x^2}}dx = \int \frac{1}{\sqrt{3^2-x^2}}dx = \arcsin\frac{x}{3} + C.$$

10. Find the indefinite integral: $\int \frac{e^{2x}}{4+e^{4x}}\,dx$.

Let $u = e^{2x}$, $du = 2e^{2x}\,dx$. Then

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{1}{2^2+(e^{2x})^2}(2e^{2x})\,dx = \frac{1}{2}\left(\frac{1}{2}\right)\arctan\frac{e^{2x}}{2}+C = \frac{1}{4}\arctan\frac{e^{2x}}{2}+C.$$

11. Evaluate the definite integral: $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx.$

Let $u = \cos x$, $du = -\sin x\,dx$. Then

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\left[\arctan(\cos x)\right]_{\pi/2}^{\pi} = -\left[\arctan(-1)-\arctan 0\right] = \frac{\pi}{4}.$$

12. Derive the formula for the derivative of the inverse tangent function.

Let $y = \arctan x \Rightarrow x = \tan y$. Differentiating implicitly, $1 = \sec^2 y\frac{dy}{dx}$. However, $\tan^2 y + 1 = \sec^2 y$, so we have $1 = (1+\tan^2 y)\frac{dy}{dx} = (1+x^2)\frac{dy}{dx}$, which gives $\frac{dy}{dx} = \frac{1}{1+x^2}$.

Lesson Twenty-Nine

1. Find the area of the region bounded by the curves $y = x^2 - 1$, $y = -x+2$, $x = 0$, and $x = 1$.

A sketch of the region shows that the line is above the parabola.

$$A = \int_0^1 \left[(-x+2)-(x^2-1)\right]dx = \int_0^1 (-x^2-x+3)dx = \left[-\frac{x^3}{3}-\frac{x^2}{2}+3x\right]_0^1 = -\frac{1}{3}-\frac{1}{2}+3 = \frac{13}{6}.$$

2. Find the area of the region bounded by the curves $y = -x^3+3$, $y = x$, $x = -1$, and $x = 1$.

A sketch of the region shows that the cubic polynomial is above the line.

$$A = \int_{-1}^1 \left[(-x^3+3)-x\right]dx = \left[-\frac{x^4}{4}+3x-\frac{x^2}{2}\right]_{-1}^1 = \left(-\frac{1}{4}+3-\frac{1}{2}\right)-\left(-\frac{1}{4}-3-\frac{1}{2}\right) = 6.$$

3. Find the area of the region bounded by the curves $y = \sqrt[3]{x-1}$ and $y = x-1$.

First find the points of intersection: $\sqrt[3]{x-1} = x-1 \Rightarrow x-1 = (x-1)^3$. Hence, $x = 1$, or $(x-1)^2 = 1 \Rightarrow x = 0,\ 2$. There are 3 points of intersection: 0, 1, and 2. By symmetry, you have

$$A = 2\int_0^1 \left[(x-1)-\sqrt[3]{x-1}\right]dx = 2\left[\frac{x^2}{2}-x-\frac{3}{4}(x-1)^{\frac{4}{3}}\right]_0^1 = 2\left[\frac{1}{2}-1+\frac{3}{4}\right] = \frac{1}{2}.$$

4. Find the area of the region bounded by the curves $x = y^2$ and $x = y+2$.

The curves intersect when $y = -1,\ 2$. We integrate with respect to the variable y, noticing that the line is to the right of the parabola.

$$A = \int_{-1}^2 \left[(y+2)-y^2\right]dy = \left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^2 = \left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right) = \frac{9}{2}.$$

5. Find the area of the region bounded by the curves $y = \cos x$ and $y = 2-\cos x$, $0 \le x \le 2\pi$.

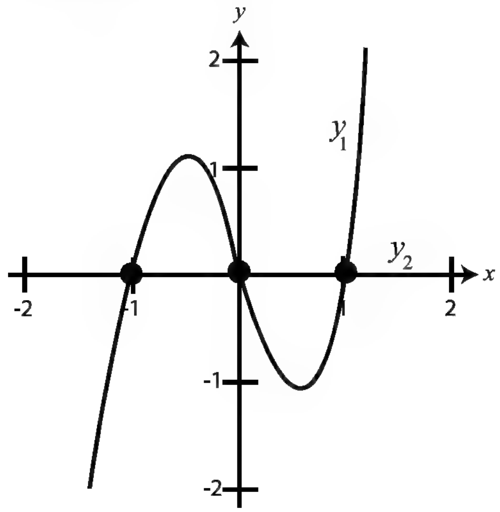
$$A = \int_0^{2\pi} \left[(2-\cos x)-\cos x\right]dx = 2\int_0^{2\pi}(1-\cos x)\,dx = 2\left[x-\sin x\right]_0^{2\pi} = 4\pi.$$

6. Find the area of the region bounded by the curves $y = xe^{-x^2}$ and $y = 0$, $0 \le x \le 1$.

$$A = \int_0^1 xe^{-x^2}dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = -\frac{1}{2}\left(e^{-1}-1\right) = \frac{1}{2}\left(1-\frac{1}{e}\right).$$

7. Find the area of the 2 regions bounded by the curves $y_1 = 3(x^3 - x)$ and $y_2 = 0$.

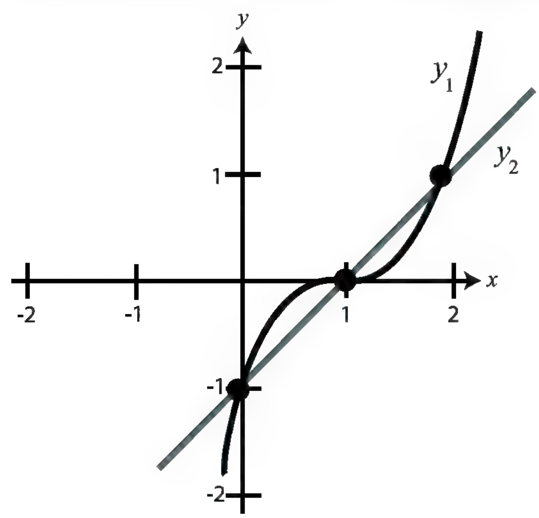
By sketching the 2 regions, you see that they have the same area (the function is odd).



Hence, $A = 2\int_{-1}^{0} 3\left(x^3 - x\right)dx = 6\int_{-1}^{0}\left(x^3 - x\right)dx = 6\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = 6\left(-\frac{1}{4} + \frac{1}{2}\right) = \frac{3}{2}.$

8. Find the area of the 2 regions bounded by the curves $y_1 = (x-1)^3$ and $y_2 = x - 1$.

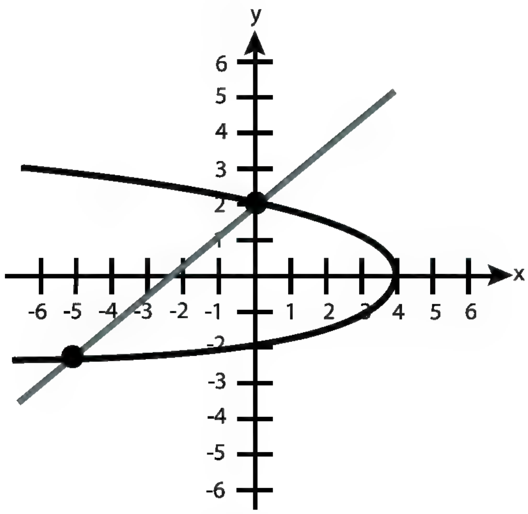
By sketching the 2 regions, you see that they have the same area.



Hence, $A = 2\int_{0}^{1}\left[(x-1)^3 - (x-1)\right]dx = 2\left[\frac{(x-1)^4}{4} - \frac{(x-1)^2}{2}\right]_{0}^{1} = 2\left(-\frac{1}{4} + \frac{1}{2}\right) = \frac{1}{2}.$

9. Consider the area bounded by $x = 4 - y^2$ and $x = y - 2$. Calculate this area by integrating with respect to x, and then with respect to y. Which method is simpler?

Setting the equations equal to each other, you see that they intersect at the points $(0,2)$ and $(-5,-3)$.



You will need 2 integrals if you integrate with respect to x:

$$\int_{-5}^{0}\left[(x+2)+\sqrt{4-x}\right]dx + \int_{0}^{4} 2\sqrt{4-x}\,dx = \frac{61}{6}+\frac{32}{3} = \frac{125}{6}.$$

The problem is easier if you integrate with respect to y: $\int_{-3}^{2}\left[\left(4-y^2\right)-(y-2)\right]dy = \frac{125}{6}.$

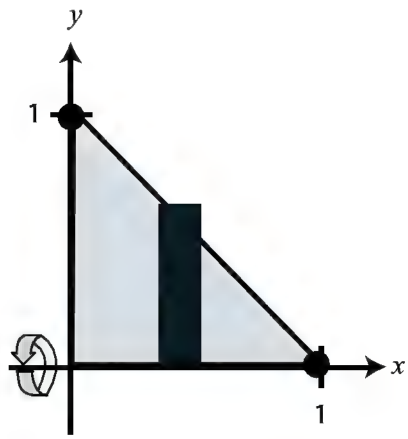
10. Set up and evaluate the definite integral that gives the area of the region bounded by the graph of $f(x) = x^3$ and its tangent line at $(1,1)$.

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$. At $(1,1)$, $f'(1) = 3$, so the equation of the tangent line is $y-1 = 3(x-1)$, or $y = 3x-2$. This tangent line intersects the cubic polynomial at $x = -2$ and $x = 1$. If you sketch the cubic and the tangent line, you will see that the cubic is above the line. Hence,

$$A = \int_{-2}^{1}\left[x^3 - (3x-2)\right]dx = \frac{27}{4}.$$

Lesson Thirty

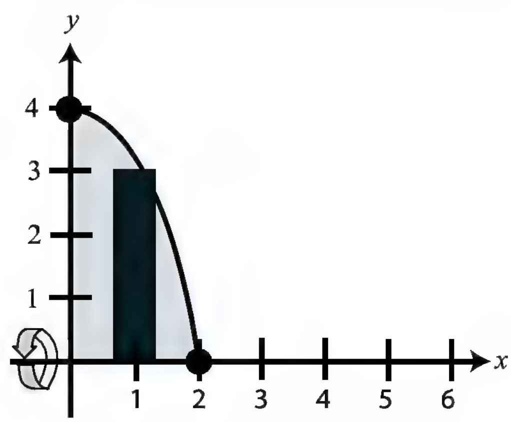
1. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations $y = -x + 1$, $y = 0$, and $x = 0$ about the x-axis.



Use the disk method with $R(x) = -x + 1$:

$$V = \pi\int_0^1 (-x+1)^2\,dx = \pi\int_0^1 (x^2 - 2x + 1)\,dx = \pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{\pi}{3}.$$

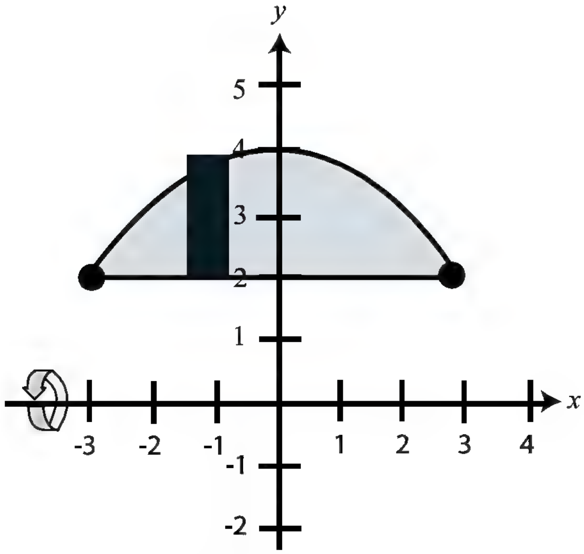
2. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations $y = 4 - x^2$, $y = 0$, and $x = 0$ about the x-axis.



Use the disk method with $R(x) = 4 - x^2$:

$$V = \pi\int_0^2 (4-x^2)^2\,dx = \pi\int_0^2 (x^4 - 8x^2 + 16)\,dx = \pi\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_0^2 = \frac{256\pi}{15}.$$

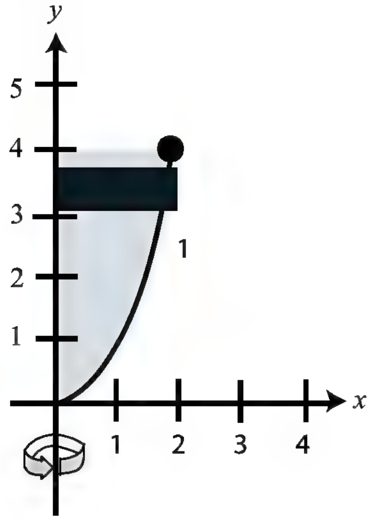
3. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations $y = 2$ and $y = 4 - \dfrac{x^2}{4}$ about the x-axis.



Solving for the points of intersection, $2 = 4 - \dfrac{x^2}{4} \Rightarrow 8 = 16 - x^2 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}.$

$$\begin{aligned}
V &= \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[\left(4 - \frac{x^2}{4} \right) - (2)^2 \right] dx \\
&= 2\pi \int_0^{2\sqrt{2}} \left[\frac{x^4}{16} - 2x^2 + 12 \right] dx \\
&= 2\pi \left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x \right]_0^{2\sqrt{2}} \\
&= 2\pi \left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2} \right] \\
&= \frac{448\sqrt{2}}{15}\pi.
\end{aligned}$$

4. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the graphs of the equations $y = x^2$, $y = 4$, and $x = 0$ about the y-axis.

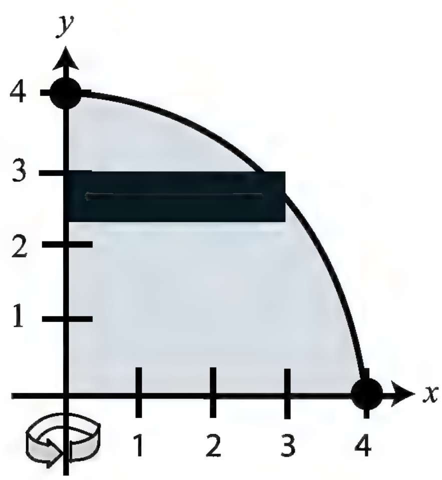


Use the disk method with $R(y) = \sqrt{y}$:

$y = x^2 \ \Rightarrow \ x = \sqrt{y}$.

$$V = \pi\int_0^4 \left(\sqrt{y}\right)^2 dy = \pi\int_0^4 y\,dy = \pi\left[\frac{y^2}{2}\right]_0^4 = 8\pi.$$

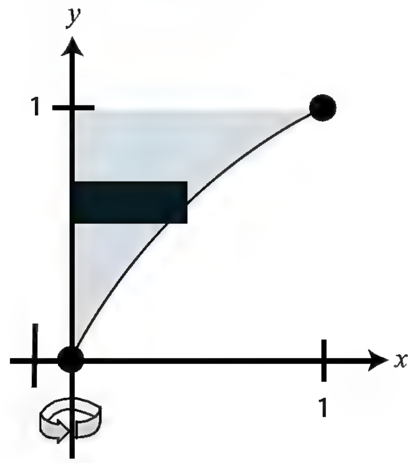
5. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the graph of the equation $y = \sqrt{16 - x^2}$ about the y-axis.



$y = \sqrt{16 - x^2} \ \Rightarrow \ x = \sqrt{16 - y^2}$.

$$V = \pi\int_0^4 \left(\sqrt{16 - y^2}\right)^2 dy = \pi\int_0^4 (16 - y^2)\,dy = \pi\left[16y - \frac{y^3}{3}\right]_0^4 = \frac{128\pi}{3}.$$

6. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the graphs of the equations $y = x^{2/3}$, $y = 1$, and $x = 0$ about the y-axis.



$y = x^{\frac{2}{3}} \Rightarrow x = y^{\frac{3}{2}}$.

$$V = \pi\int_0^1 \left(y^{\frac{3}{2}}\right)^2 dy = \pi\int_0^1 y^3\, dy = \pi\left[\frac{y^4}{4}\right]_0^1 = \frac{\pi}{4}.$$

7. Find the volume of the solid generated by revolving the region bounded by the curves $y = \sec x$ and $y = 0$, $0 \le x \le \frac{\pi}{3}$, about the line $y = 4$.

$$\begin{aligned} V &= \pi\int_0^{\pi/3}\left[(4)^2 - (4-\sec x)^2\right]dx \\ &= \pi\int_0^{\pi/3}\left(8\sec x - \sec^2 x\right)dx \\ &= \pi\left[8\ln\left|\sec x + \tan x\right| - \tan x\right]_0^{\pi/3} \\ &= \pi\left[\left(8\ln\left|2+\sqrt{3}\right| - \sqrt{3}\right) - \left(8\ln\left|1+0\right| - 0\right)\right] \\ &= \pi\left(8\ln\left|2+\sqrt{3}\right| - \sqrt{3}\right) \approx 27.66. \end{aligned}$$

8. Find the volume of the solid generated by revolving the region bounded by the curves $y = 5 - x$, $y = 0$, $y = 4$, and $x = 0$ about the line $x = 5$.

If you sketch the region, you see that the outer radius is $R(y) = 5$ and the inner radius is $r(y) = 5 - (5 - y) = y$. Hence the volume is

$$V = \pi\int_0^4 \left[5^2 - y^2\right]dy = \pi\left[25y - \frac{y^3}{3}\right]_0^4 = \pi\left[100 - \frac{64}{3}\right] = \frac{236\pi}{3}.$$

9. Use the disk method to verify that the volume of a sphere is $\frac{4}{3}\pi r^3$.

Sketch the semicircle of radius r, $y = \sqrt{r^2 - x^2}$. Revolve this region about the x-axis to generate the sphere:

$$V = \pi\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx = 2\pi\int_0^r \left(r^2 - x^2\right)dx = 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3.$$

10. A sphere of radius r is cut by a plane h $(h < r)$ units above the equator. Find the volume of the solid (spherical segment) above the plane.

Sketch a circle of radius r and the horizontal line $y = h,\ 0 < h < r$.

Integrate with respect to y: $R(y) = \sqrt{r^2 - y^2}$.

Revolve this region about the y-axis:

$$\begin{aligned} V &= \pi\int_h^r \left(\sqrt{r^2 - y^2}\right)^2 dy \\ &= \pi\int_h^r \left(r^2 - y^2\right) dy \\ &= \pi\left[r^2 y - \frac{y^3}{3}\right]_h^r \\ &= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2 h - \frac{h^3}{3}\right)\right] \\ &= \frac{\pi}{3}\left(2r^3 - 3r^2 h + h^3\right). \end{aligned}$$

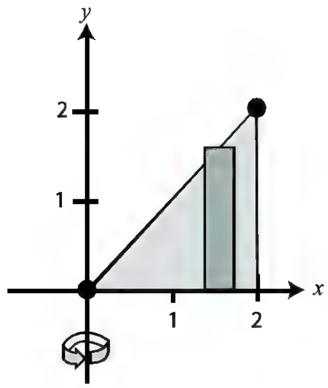
Note that if $h = 0$, you obtain the volume of a hemisphere.

11. Describe the volume represented by the definite integral: $\pi\int_2^4 y^4\, dy$.

Note that $R(y) = y^2$. This integral represents the volume of the solid generated by revolving the region bounded by $x = y^2,\ x = 0,\ y = 2,$ and $y = 4$ about the y-axis.

Lesson Thirty-One

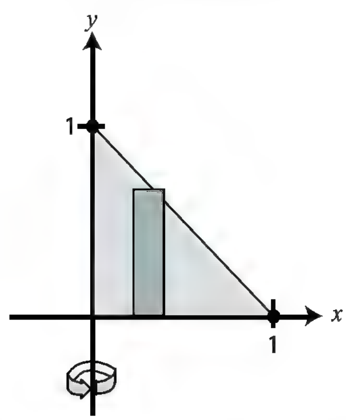
1. The region in the first quadrant bounded by the graphs of the equations $y = x$ and $x = 2$ is revolved about the y-axis. Use the shell method to find the volume of the resulting solid.



The radius of the shell is x, and the height is x.

$$V = 2\pi\int_0^2 x(x)\,dx = 2\pi\left[\frac{x^3}{3}\right]_0^2 = \frac{16\pi}{3}.$$

2. The region in the first quadrant bounded by the graph of the equation $y = 1 - x$ is revolved about the y-axis. Use the shell method to find the volume of the resulting solid.



The radius of the shell is x, and the height is $1 - x$.

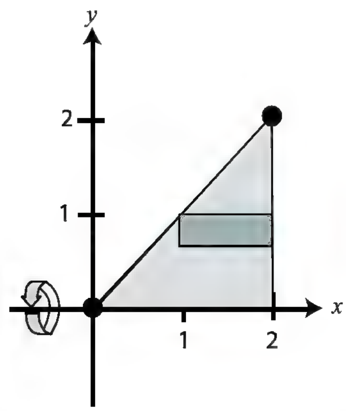
$$V = 2\pi\int_0^1 x(1-x)\,dx = 2\pi\int_0^1 (x - x^2)\,dx = 2\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}.$$

3. The region in the first quadrant bounded by the graphs of the equations $y = x^2$ and $x = 3$ is revolved about the y-axis. Use the shell method to find the volume of the resulting solid.

The radius of the shell is x, and the height is x^2.

$$V = 2\pi\int_0^3 x(x^2)\,dx = 2\pi\left[\frac{x^4}{4}\right]_0^3 = \frac{81\pi}{2}.$$

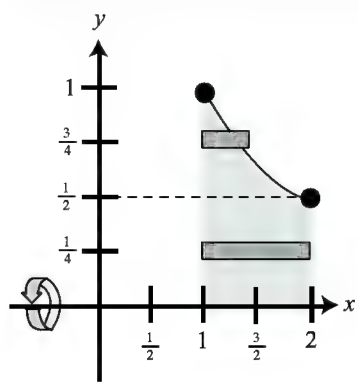
4. The region in the first quadrant bounded by the graphs of the equations $y = x$ and $x = 2$ is revolved about the x-axis. Use the shell method to find the volume of the resulting solid.



The radius of the shell is y, and its height is $2 - y$.

$$V = 2\pi\int_0^2 y(2-y)\,dy = 2\pi\int_0^2 (2y - y^2)\,dy = 2\pi\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{8\pi}{3}.$$

5. The region in the first quadrant bounded by the graphs of the equations $y = \frac{1}{x}$, $x = 1$, and $x = 2$ is revolved about the x-axis. Use the shell method to find the volume of the resulting solid.



For $0 \le y < \frac{1}{2}$, the radius of the shell is y, and the height is 1. For $\frac{1}{2} \le y \le 1$, the radius of the shell is y, and the height is $\frac{1}{y} - 1$.

$$V = 2\pi\int_0^{1/2} y\,dy + 2\pi\int_{1/2}^{1} (1-y)\,dy = 2\pi\left[\frac{y^2}{2}\right]_0^{1/2} + 2\pi\left[y - \frac{y^2}{2}\right]_{1/2}^{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

6. The region in the first quadrant bounded by the graph of the equation $y = 4x - x^2$ is revolved about the line $x = 5$. Use the shell method to find the volume of the resulting solid.

The radius of the shell is $5 - x$, and the height is $4x - x^2$.

$$V = 2\pi\int_0^4 (5-x)(4x - x^2)\,dx = 2\pi\int_0^4 \left(x^3 - 9x^2 + 20x\right)dx = 2\pi\left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_0^4 = 64\pi.$$

7. Find the volume of the torus formed by revolving the region bounded by the circle $x^2+y^2=r^2$ about the line $x=R$ $(r<R)$.

Sketch the circle and the vertical line $x=R$. We will find the volume of the upper half and double the answer. The radius of the shell is $R-x$, and the height is $\sqrt{r^2-x^2}$.

$$V = 4\pi\int_{-r}^{r}(R-x)\sqrt{r^2-x^2}\,dx = 4\pi R\int_{-r}^{r}\sqrt{r^2-x^2}\,dx - 4\pi\int_{-r}^{r}x\sqrt{r^2-x^2}\,dx.$$

The first integral is the area of a semicircle, and the second integral is 0 because the integrand is an odd function. Hence, $V=4\pi R\left(\frac{\pi r^2}{2}\right)=2\pi^2r^2R$.

8. Describe the volume represented by the definite integral: $2\pi\int_0^2 x^3\,dx$.

$2\pi\int_0^2 x^3\,dx = 2\pi\int_0^2 x(x^2)\,dx$ represents the volume obtained by revolving about the y-axis the region bounded by $y=x^2$, $y=0$, $x=0$, and $x=2$. Note: Other answers are possible.

9. Describe the volume represented by the definite integral: $2\pi\int_0^1\left(y-y^{\frac{3}{2}}\right)dy$.

$2\pi\int_0^1\left(y-y^{\frac{3}{2}}\right)dy = 2\pi\int_0^1 y\left(1-\sqrt{y}\right)dy$ represents the volume obtained by revolving about the x-axis the region bounded by $x=\sqrt{y}$, $x=1$, and $y=0$. Note: Other answers are possible.

Lesson Thirty-Two

1. Find the distance between the points $(0,0)$ and $(8,15)$ using the distance formula and the arc length formula.

The distance formula: $d=\sqrt{(8-0)^2+(15-0)^2}=\sqrt{64+225}=\sqrt{289}=17$.

The equation of the line segment joining the points is $y=\frac{15}{8}x,\ 0\le x\le 8$. Since $y'=\frac{15}{8}$, the arc length formula tells us the length is

$$s = \int_0^8\sqrt{1+\left(\frac{15}{8}\right)^2}\,dx = \int_0^8\sqrt{\frac{64+225}{64}}\,dx = \int_0^8\sqrt{\frac{289}{64}}\,dx = \int_0^8\frac{17}{8}dx = \left[\frac{17x}{8}\right]_0^8 = 17.$$

2. Find the arc length of the graph of the function $y=\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$ over the interval $0\le x\le 1$.

$y=\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$.

$y'=\left(x^2+1\right)^{\frac{1}{2}}(2x)$.

$1+(y')^2=1+4x^2(x^2+1)=4x^4+4x^2+1=\left(2x^2+1\right)^2$.

$$s = \int_0^1\sqrt{1+(y')^2}\,dx = \int_0^1\left(2x^2+1\right)dx = \left[\frac{2x^3}{3}+x\right]_0^1 = \frac{5}{3}.$$

3. Find the arc length of the graph of the function $y=\frac{2}{3}x^{\frac{3}{2}}+1$ over the interval $0\le x\le 1$.

$y=\frac{2}{3}x^{\frac{3}{2}}+1$.

$y'=x^{\frac{3}{2}}$.

$1+(y')^2=1+x$.

$$s = \int_0^1\sqrt{1+(y')^2}\,dx = \int_0^1\sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left(\sqrt{8}-1\right).$$

4. Find the arc length of the graph of the function $y=\frac{x^4}{8}+\frac{1}{4x^2}$ over the interval $1\le x\le 3$.

$y=\frac{x^4}{8}+\frac{1}{4x^2}$.

$y'=\frac{x^3}{2}-\frac{1}{2x^3}$.

$1+(y')^2=1+\left(\frac{x^3}{2}-\frac{1}{2x^3}\right)^2=\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)^2$.

$$s = \int_1^3\sqrt{1+(y')^2}\,dx = \int_1^3\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)dx = \left[\frac{x^4}{8}-\frac{1}{4x^2}\right]_1^3 = \frac{92}{9}.$$

5. Find the arc length of the graph of the function $y=\frac{1}{2}\left(e^{x}+e^{-x}\right)$ over the interval $0\le x\le 2$.

$y=\frac{1}{2}\left(e^{x}+e^{-x}\right).$

$y'=\frac{1}{2}\left(e^{x}-e^{-x}\right).$

$1+(y')^2=\left[\frac{1}{2}\left(e^{x}+e^{-x}\right)\right]^2.$

$$s \;=\; \int_0^2\sqrt{1+(y')^2}\,dx \;=\; \int_0^2\frac{1}{2}\left(e^{x}+e^{-x}\right)dx \;=\; \frac{1}{2}\left[e^{x}-e^{-x}\right]_0^2 \;=\; \frac{1}{2}\left(e^{2}-\frac{1}{e^{2}}\right).$$

6. Set up the integral for the arc length of the graph of the function $y=\sin x$ over the interval $0\le x\le \pi$. Use a graphing utility to approximate this integral.

$y=\sin x.$

$y'=\cos x.$

$1+(y')^2=1+\cos^2 x.$

$$s \;=\; \int_0^{\pi}\sqrt{1+(y')^2}\,dx \;=\; \int_0^{\pi}\sqrt{1+\cos^2 x}\,dx \;\approx\; 3.820.$$

7. Set up and evaluate the definite integral for the area of the surface generated by revolving $y=\frac{x}{2},\ 0\le x\le 6,$ about the x-axis.

$y=\frac{x}{2}.$

$y'=\frac{1}{2}.$

$1+(y')^2=\frac{5}{4}.$

$$S \;=\; 2\pi\int_0^6\frac{x}{2}\sqrt{\frac{5}{4}}\,dx \;=\; \left[\frac{\pi\sqrt{5}}{4}x^2\right]_0^6 \;=\; 9\pi\sqrt{5}.$$

8. Set up and evaluate the definite integral for the area of the surface generated by revolving $y=\sqrt{4-x^2},\ -1\le x\le 1$, about the x-axis.

$y=\sqrt{4-x^2}.$

$y'=\frac{-x}{\sqrt{4-x^2}}.$

$1+(y')^2=1+\frac{x^2}{4-x^2}=\frac{4}{4-x^2}.$

$$S \;=\; 2\pi\int_{-1}^{1}\sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}\,dx \;=\; 2\pi\int_{-1}^{1}2\,dx \;=\; 4\pi\left[x\right]_{-1}^{1} \;=\; 8\pi.$$

9. Set up and evaluate the definite integral for the area of the surface generated by revolving $y = 9 - x^2,\ 0 \le x \le 3$, about the y-axis.

$y = 9 - x^2.$
$y' = -2x.$
$1 + (y')^2 = 1 + 4x^2.$

$$S \;=\; 2\pi\int_0^3 x\sqrt{1+4x^2}\,dx \;=\; \frac{\pi}{4}\int_0^3 \left(1+4x^2\right)^{\frac{1}{2}}(8x)\,dx \;=\; \left[\frac{\pi}{6}\left(1+4x^2\right)^{\frac{3}{2}}\right]_0^3 \;=\; \frac{\pi}{6}\left(37^{\frac{3}{2}}-1\right).$$

10. Set up and evaluate the definite integral for the area of the surface generated by revolving $y = 2x + 5,\ 1 \le x \le 4$, about the y-axis.

$y = 2x + 5.$
$y' = 2.$

$$S \;=\; 2\pi\int_1^4 x\sqrt{1+4}\,dx \;=\; 2\pi\sqrt{5}\left[\frac{x^2}{2}\right]_1^4 \;=\; 15\sqrt{5}\pi.$$

Lesson Thirty-Three

1. Evaluate the integral: $\int \frac{9}{(t-8)^2}\,dt$.

Let $u = t-8,\ du = dt$. Then $\int \frac{9}{(t-8)^2}\,dt \ = \ 9\int (t-8)^{-2}\,dt \ = \ \frac{-9}{t-8} + C.$

2. Evaluate the integral: $\int \frac{t^2-3}{-t^3+9t+1}\,dt$.

Let $u = -t^3+9t+1,\ du = (-3t^2+9)\,dt = -3(t^2-3)\,dt.$ Then

$$\int \frac{t^2-3}{-t^3+9t+1}\,dt \ = \ \frac{-1}{3}\int \frac{-3(t^2-3)}{-t^3+9t+1}\,dt \ = \ \frac{-1}{3}\ln\left|-t^3+9t+1\right| + C.$$

3. Evaluate the integral: $\int \frac{e^x}{1+e^x}\,dx$.

Let $u = 1+e^x,\ du = e^x\,dx$. Then $\int \frac{e^x}{1+e^x}\,dx \ = \ \ln(1+e^x) + C.$

4. Evaluate the integral: $\int \frac{x^2}{x-1}\,dx$.

By long division, $\int \frac{x^2}{x-1}\,dx \ = \ \int (x+1)\,dx \ + \ \int \frac{1}{x-1}\,dx \ = \ \frac{x^2}{2} + x + \ln|x-1| + C.$

5. Evaluate the integral: $\int \frac{1+\sin x}{\cos x}\,dx$.

$$\int \frac{1+\sin x}{\cos x}\,dx \ = \ \int (\sec x + \tan x)\,dx \ = \ \ln|\sec x + \tan x| - \ln|\cos x| + C.$$

6. Evaluate the integral: $\int \frac{1}{\cos\theta - 1}\,d\theta$.

$$\begin{aligned}
\int \frac{1}{\cos\theta-1}\,d\theta \ &= \ \int \left(\frac{1}{\cos\theta-1}\right)\left(\frac{\cos\theta+1}{\cos\theta+1}\right)d\theta \\
&= \ \int \frac{\cos\theta+1}{\cos^2\theta-1}\,d\theta \\
&= \ \int \frac{\cos\theta+1}{-\sin^2\theta}\,d\theta \\
&= \ \int \left(-\csc\theta\cot\theta - \csc^2\theta\right)d\theta \\
&= \ \csc\theta + \cot\theta + C \\
&= \ \frac{1+\cos\theta}{\sin\theta} + C.
\end{aligned}$$

7. Evaluate the integral: $\int \frac{6}{\sqrt{10x-x^2}}dx$.

Note that $10x-x^2=25-(25-10x+x^2)=25-(5-x)^2$.

$$\int \frac{6}{\sqrt{10x-x^2}}dx = 6\int \frac{1}{\sqrt{25-(5-x)^2}}dx = -6\int \frac{-1}{\sqrt{5^2-(5-x)^2}}dx = -6\arcsin\frac{5-x}{5}+C =$$

$$6\arcsin\frac{x-5}{5}+C.$$

8. Solve the differential equation $\left(4+\tan^2 x\right)y'=\sec^2 x$.

The equation can be rewritten as $y'=\frac{\sec^2 x}{4+\tan^2 x}$. Let $u=\tan x,\ du=\sec^2 x\,dx$. Then

$$y = \int \frac{\sec^2 x}{4+\tan^2 x}dx = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right)+C.$$

9. Evaluate the definite integral: $\int_0^8 \frac{2x}{\sqrt{x^2+36}}dx$.

Let $u=x^2+36,\ du=2xdx$. Then

$$\int_0^8 \frac{2x}{\sqrt{x^2+36}}dx = \int_0^8 \left(x^2+36\right)^{-\frac{1}{2}}(2x)dx = 2\left[\left(x^2+36\right)^{\frac{1}{2}}\right]_0^8 = 2(10-6) = 8.$$

10. Evaluate the definite integral: $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}dx$.

Let $u=3x,\ du=3\,dx$. Then $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}dx = \frac{1}{3}\int_0^{2/\sqrt{3}} \frac{3}{2^2+(3x)^2}dx = \left[\frac{1}{6}\arctan\left(\frac{3x}{2}\right)\right]_0^{2/\sqrt{3}} = \frac{\pi}{18}.$

11. Explain why the antiderivative $y_1=e^{x+C_1}$ is equivalent to the antiderivative $y_2=Ce^x$.

You can write $y_1=e^{x+C_1}=e^x e^{C_1}=e^x C=Ce^x$, where e^{C_1} is a constant.

12. Find the arc length of the graph of $y=\ln(\sin x)$ from $x=\frac{\pi}{4}$ to $x=\frac{\pi}{2}$.

$y=\ln(\sin x)$.

$y'=\frac{\cos x}{\sin x}=\cot x.$

$1+(y')^2=1+\cot^2 x=\csc^2 x.$

$$s = \int_{\pi/4}^{\pi/2}\csc x\,dx = \left[-\ln\left|\csc x+\cot x\right|\right]_{\pi/4}^{\pi/2} = -\ln 1+\ln(\sqrt{2}+1) = \ln(\sqrt{2}+1).$$

13. Show that the 2 answers to Example 2 are equivalent. That is, show that $\tan x + \sec x + C = \dfrac{-\cos x}{\sin x - 1} + C_1$.

$$\begin{aligned}
\tan x + \sec x + C &= \frac{\sin x}{\cos x} + \frac{1}{\cos x} + C \\
&= \left(\frac{\sin x + 1}{\cos x}\right)\left(\frac{\sin x - 1}{\sin x - 1}\right) + C \\
&= \frac{\sin^2 x - 1}{\cos x(\sin x - 1)} + C \\
&= \frac{-\cos^2 x}{\cos x(\sin x - 1)} + C_1 \\
&= \frac{-\cos x}{\sin x - 1} + C_1.
\end{aligned}$$

Lesson Thirty-Four

1. Evaluate $\int x\sin 3x\,dx$ using integration by parts with $u = x$ and $dv = \sin 3x\,dx$.

$du = dx,\ v = \frac{-1}{3}\cos 3x.$

$$\int x\sin 3x\,dx \;=\; uv - \int v\,du \;=\; x\left(\frac{-1}{3}\cos 3x\right) \;-\; \int \frac{-1}{3}\cos 3x\,dx \;=\; \frac{-x}{3}\cos 3x \;+\; \frac{1}{9}\sin 3x \;+\; C.$$

2. Evaluate $\int x^3 \ln x\,dx$ using integration by parts with $u = \ln x$ and $dv = x^3\,dx$.

$du = \frac{1}{x}dx, v = \frac{x^4}{4}.$

$$\int x^3 \ln x\,dx \;=\; uv \;-\; \int v\,du \;=\; (\ln x)\frac{x^4}{4} \;-\; \int\left(\frac{x^4}{4}\right)\left(\frac{1}{x}\right)dx \;=\; \frac{x^4}{4}\ln x \;-\; \int \frac{x^3}{4}dx \;=\; \frac{x^4}{4}\ln x \;-\; \frac{x^4}{16} \;+\; C.$$

3. Evaluate the integral: $\int xe^{-4x}\,dx$.

$u = x,\ dv = e^{-4x}\,dx,\ du = dx,$ and $v = \frac{-e^{-4x}}{4}.$

$$\int xe^{-4x}\,dx \;=\; uv \;-\; \int v\,du \;=\; x\left(\frac{-e^{-4x}}{4}\right) \;-\; \int \frac{-e^{-4x}}{4}dx \;=\; \frac{-xe^{-4x}}{4} \;-\; \frac{e^{-4x}}{16} \;+\; C.$$

4. Evaluate the integral: $\int x\cos x\,dx$.

$u = x,\ dv = \cos x\,dx,\ du = dx,$ and $v = \sin x.$

$$\int x\cos x\,dx \;=\; uv \;-\; \int v\,du \;=\; x\sin x \;-\; \int \sin x dx \;=\; x\sin x \;+\; \cos x \;+\; C.$$

5. Evaluate the integral: $\int \arctan x\,dx$.

$u = \arctan x,\ dv = dx,\ du = \frac{1}{1+x^2}dx,$ and $v = x.$

$$\int \arctan x\,dx \;=\; uv \;-\; \int v\,du \;=\; \arctan x(x) \;-\; \int x\left(\frac{1}{1+x^2}\right)dx \;=\; x\arctan x \;-\; \frac{1}{2}\ln(1+x^2) \;+\; C.$$

6. Evaluate the integral: $\int \frac{\ln 2x}{x^2}dx$.

$u = \ln 2x,\ dv = \frac{1}{x^2}dx,\ du = \frac{1}{x}dx,$ and $v = \frac{-1}{x}.$

$$\int \frac{\ln 2x}{x^2}dx \;=\; uv \;-\; \int v\,du \;=\; \ln 2x\left(\frac{-1}{x}\right) \;-\; \int\left(\frac{-1}{x}\right)\left(\frac{1}{x}\right)dx \;=\; \frac{-1}{x}\ln 2x \;+\; \int x^{-2}dx \;=\; \frac{-1}{x}\ln 2x \;-\; \frac{1}{x} + C.$$

7. Use substitution and integration by parts to evaluate the integral: $\int \sin\sqrt{x}\,dx$.

Let $w=\sqrt{x},\ w^2=x,$ and $2wdw=dx$. Then $\int \sin\sqrt{x}\,dx=\int \sin w 2wdw=2\int w\sin wdw$.

Now use integration by parts: $u=w,\ dv=\sin w(dw),\ du=dw,$ and $v=-\cos w.$

$$\int w\sin wdw = uv - \int vdu = w(-\cos w) - \int -\cos wdw = -w\cos w + \sin w + C.$$

Now go back to the original variable to obtain the final answer:

$$\int \sin\sqrt{x}\,dx = 2\int w\sin wdw = 2[-w\cos w+\sin w] + C = 2\left[-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x}\right] + C.$$

8. Evaluate the integral: $\int \sin^7 2x\cos 2x\,dx$.

Let $u=\sin 2x,\ du=2\cos 2x\,dx.$

$$\int \sin^7 2x(\cos 2x)\,dx = \frac{1}{2}\int \sin^7 2x(2\cos 2x)dx = \frac{1}{2}\left(\frac{\sin^8 2x}{8}\right) + C = \frac{\sin^8 2x}{16} + C.$$

9. Evaluate the integral: $\int \sin^3 x\cos^2 x\,dx$.

$$\begin{aligned}\int \sin^3 x(\cos^2 x)\,dx &= \int (1-\cos^2 x)\cos^2 x(\sin x)dx\\ &= -\int\left(\cos^2 x-\cos^4 x\right)(-\sin x)\,dx\\ &= -\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C.\end{aligned}$$

10. Evaluate the integral: $\int \sec^2 x\tan x\,dx$.

Let $u=\tan x,\ du=\sec^2 x\,dx$. Then $\int \sec^2 x\tan x\,dx = \frac{1}{2}\tan^2 x + C.$

Note: You could also let $u=\sec x,\ du=\sec x\tan x\,dx.$

11. Evaluate the integral: $\int \tan^2 x\sec^4 x\,dx$.

$$\int \tan^2 x\sec^4 x\,dx = \int \tan^2 x(\tan^2 x+1)\sec^2 x\,dx = \int\left(\tan^4 x+\tan^2 x\right)\sec^2 x\,dx = \frac{\tan^5 x}{5}+\frac{\tan^3 x}{3}+C.$$

12. Evaluate the integral: $\int \cos^2 x\,dx$.

We use the half-angle formula, $\cos^2 x=\dfrac{1+\cos 2x}{2}$.

$$\int \cos^2 x\,dx = \int\frac{1+\cos 2x}{2}dx = \frac{1}{2}x+\frac{\sin 2x}{4}+C.$$

13. Which integral is easier to evaluate, $\int \tan^{400} x\sec^2 x\,dx$ or $\int \tan^{400} x\sec x\,dx$? Why?

The first integral is easier because if $u=\tan x,$ then $du=\sec^2 x\,dx$.

Lesson Thirty-Five

1. Verify that $y = Ce^{4x}$ is a solution to the differential equation $y' = 4y$.

$y = Ce^{4x}$, $y' = 4Ce^{4x}$. Substituting into the differential equation,

$y' = 4Ce^{4x} = 4(Ce^{4x}) = 4y$.

2. Verify that $y = C_1 \sin x + C_2 \cos x$ is a solution to the differential equation $y'' + y = 0$.

$y = C_1 \sin x + C_2 \cos x,\ y' = C_1 \cos x - C_2 \sin x,$ and $y'' = -C_1 \sin x - C_2 \cos x$.

Substituting into the differential equation,

$y'' + y = (-C_1 \sin x - C_2 \cos x) + (C_1 \sin x + C_2 \cos x) = 0$.

3. Verify that $y = 4e^{-6x^2}$ is a solution to the differential equation $y' = -12xy,\ y(0) = 4$.

$y = 4e^{-6x^2}$, and $y' = 4e^{-6x^2}(-12x) = -48xe^{-6x^2}$. Substituting into the differential equation,

$y' = -48xe^{-6x^2} = -12(x)4e^{-6x^2} = -12xy$. Also, $y(0) = 4e^0 = 4$.

4. Use integration to find a general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x}{1+x^2}$.

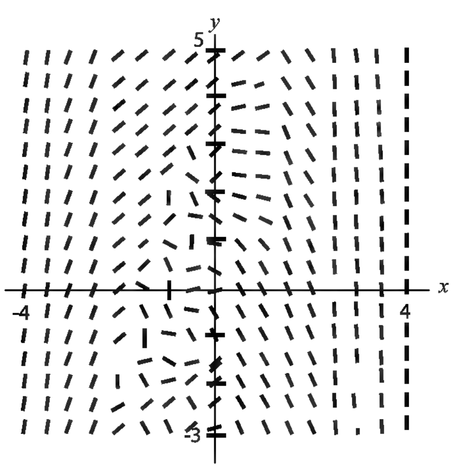
$$y = \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2) + C.$$

5. Use integration to find a general solution of the differential equation $\dfrac{dy}{dx} = x\sqrt{x-6}$.

Let $u = \sqrt{x-6}$. Thus $u^2 = x-6,\ x = u^2+6,$ and $dx = 2u\,du.$ Then we have

$$\begin{aligned}
y &= \int x\sqrt{x-6}\,dx \\
&= \int \left(u^2+6\right)(u)(2u\,du) \\
&= 2\int \left(u^4+6u^2\right)du \\
&= 2\left[\frac{u^5}{5} + 2u^3\right] + C \\
&= \frac{2}{5}(x-6)^{\frac{5}{2}} + 4(x-6)^{\frac{3}{2}} + C.
\end{aligned}$$

6. Sketch a slope field for the differential equation $y' = y - 4x$.



7. Use separation of variables to solve the differential equation $\frac{dy}{dx}=x+3$.

$\frac{dy}{dx}=x+3 \Rightarrow dy=(x+3)\,dx \Rightarrow \int dy = \int (x+3)\,dx$. Hence $y=\frac{x^2}{2}+3x+C$.

8. Use separation of variables to solve the differential equation $\frac{dy}{dx}=\frac{5x}{y}$.

$\frac{dy}{dx}=\frac{5x}{y} \Rightarrow y\,dy=5x\,dx \Rightarrow \int y\,dy = \int 5x\,dx$. Hence $\frac{y^2}{2}=\frac{5x^2}{2}+C_1$, which can be rewritten as $y^2-5x^2=C$.

9. Use separation of variables to solve the differential equation $\frac{dy}{dx}=x(1+y)$.

$\frac{dy}{dx}=x(1+y) \Rightarrow \frac{1}{1+y}dy=x\,dx \Rightarrow \int \frac{1}{1+y}dy = \int x\,dx$. Hence $\ln|1+y|=\frac{x^2}{2}+C_1$, which can be rewritten as $y=Ce^{x^2/2}-1$.

10. Use separation of variables to solve the differential equation $\frac{dy}{dt}=ky$, where y is a function of t and k is a constant.

$\frac{dy}{dt}=ky \Rightarrow \frac{dy}{y}=k\,dt \Rightarrow \int \frac{dy}{y} = \int k\,dt$. Hence $\ln|y|=kt+C_1 \Rightarrow y=e^{kt+C_1}=Ce^{kt}$.

Lesson Thirty-Six

1. The rate of change of y is proportional to y. When $x=0,\ y=6,$ and when $x=4,\ y=15$. Write and solve the differential equation that models this verbal statement. Then find the value of y when $x=8$.

The differential equation is $\dfrac{dy}{dx}=ky$ with solution $y=Ce^{kx}$. When $x=0,\ y=6=Ce^{0}\ \Rightarrow\ C=6$. When $x=4,\ y=15=6e^{k(4)}\ \Rightarrow\ e^{4k}=\dfrac{5}{2}\ \Rightarrow\ k=\dfrac{1}{4}\ln\dfrac{5}{2}$. Hence, $y=6e^{\left[\frac{1}{4}\ln(5/2)\right]x}=6\left(\dfrac{5}{2}\right)^{\frac{x}{4}}\approx 6e^{0.2291x}$. At $x=8$, $y=6e^{\left[\frac{1}{4}\ln(5/2)\right]8}=6e^{\ln\left(\frac{5}{2}\right)^2}=6\left(\dfrac{25}{4}\right)=\dfrac{75}{2}$.

2. The rate of change of N is proportional to N. When $t=0,\ N=250,$ and when $t=1,\ N=400$. Write and solve the differential equation that models this verbal statement. Then find the value of N when $t=4$.

The differential equation is $\dfrac{dN}{dt}=kN$ with solution $N=Ce^{kt}$. When $t=0$, $N=250=Ce^{k(0)}\ \Rightarrow\ C=250.$ When $t=1,\ N=400=250e^{k(1)}\ \Rightarrow\ e^{k}=\dfrac{8}{5}\ \Rightarrow\ k=\ln\dfrac{8}{5}.$ Hence, $N=250e^{t\ln(8/5)}=250\left(\dfrac{8}{5}\right)^{t}\approx 250e^{0.4700t}$. When $t=4$, $N=250e^{4\ln(8/5)}\approx 250e^{\ln(8/5)^4}=250\left(\dfrac{8}{5}\right)^4=\dfrac{8192}{5}$.

3. Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?

$y=Ce^{kt},\ \dfrac{1}{2}C=Ce^{k(1599)}\ \Rightarrow\ k=\dfrac{1}{1599}\ln\dfrac{1}{2}$. When $t=100$, $y=Ce^{[\ln(1/2)/1599](100)}\approx 0.9576C$. Therefore, about 95.76% remains after 100 years.

4. The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours. Find the initial population. Also, how many bacteria will there be after 8 hours?

$y=Ce^{kt}$. At time $t=2,\ 125=Ce^{k(2)}\ \Rightarrow\ C=125e^{-2k}$. At time $t=4,\ 350=Ce^{k(4)}$.

Hence, $350=\left(125e^{-2k}\right)e^{4k}=125e^{2k}\ \Rightarrow\ e^{2k}=\dfrac{14}{5}\ \Rightarrow\ k=\dfrac{1}{2}\ln\dfrac{14}{5}$. So

$C=125e^{-2k}=125e^{-2(1/2)\ln(14/5)}=125\left(\dfrac{5}{14}\right)=\dfrac{625}{14}$, which is the initial population. (Of course this is just an approximation, since the number of bacteria must be a positive integer.) After 8 hours, there will be $y=\dfrac{625}{14}e^{(1/2)\ln(14/5)8}=\dfrac{625}{14}\left(\dfrac{14}{5}\right)^4=2744$ bacteria.

5. Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after another 2 months?

Let $y=Ce^{kt}$, where t is measured in months. From the initial condition, $C=100{,}000$. Because $y=80{,}000$ when $t=4$, you have $80{,}000=100{,}000e^{4k}\ \Rightarrow\ 0.8=e^{4k}\ \Rightarrow\ k=\dfrac{1}{4}\ln(0.8)\approx -0.0558$. So after 2 more months ($t=6$), the monthly sales will be $y\approx 100{,}000e^{-0.0558(6)}\approx 71{,}500$ units.

6. When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F, its core temperature is 1500°F. One hour after it is removed, the core temperature is 1120°F. Find the core temperature 5 hours after the object is removed from the furnace.

The differential equation for Newton's law of cooling is $\frac{dy}{dt} = k(y-80)$.

Separating variables and integrating, we have $\frac{dy}{y-80} = k\,dt \Rightarrow \int \frac{dy}{y-80} = \int k\,dt \Rightarrow \ln(y-80) = kt + C$.

When $t = 0$, $y = 1500 \Rightarrow \ln(1500-80) = \ln 1420 = C$.

When $t = 1$, $y = 1120 \Rightarrow \ln(1120-80) = k(1) + \ln 1420$.

Solving for k, you obtain $k = \ln 1040 - \ln 1420 = \ln \frac{1040}{1420} = \ln \frac{52}{71}$.

So the model is $\ln(y-80) = kt + C \Rightarrow y - 80 = e^{kt+C} = e^{kt}e^{C} \Rightarrow y = 80 + 1420e^{[\ln(52/71)t]}$.

Finally, when $t = 5$, $y = 80 + 1420e^{[\ln(52/71)5]} \approx 379.2$°F.

Summary Sheet

Note: The number in parentheses indicates the lesson in which the concept or term is introduced.

arc length formula (32): Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a,b]$. The arc length of f between a and b is $s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$.

chain rule (10): If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x, and $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$; or equivalently, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

compound interest formula (27): Let P be the amount of a deposit at an annual interest rate of r (as a decimal) compounded n times per year. The amount after t years is $A = P\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, the amount is $A = Pe^{rt}$.

continuity (5): A function f is continuous at c if the following 3 conditions are met:

$f(c)$ is defined,

$\lim_{x\to c} f(x)$ exists, and

$\lim_{x\to c} f(x) = f(c)$.

critical number (12): Let f be defined at c. If $f'(c) = 0$ or f is not differentiable at c, then c is a critical number of f.

derivative (7): The derivative of f at x is given by the following limit, if it exists:

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

derivatives of the sine and cosine functions (8):

$$\frac{d}{dx}[\sin x] = \cos x.$$

$$\frac{d}{dx}[\cos x] = -\sin x.$$

free-falling object (8, 18): The position s of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation $s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$, where t is time, s_0 is the initial height, v_0 is the initial velocity, and g is the acceleration due to gravity. On Earth, $g \approx -32$ ft/sec^2 ≈ -9.8 m/sec^2.

horizontal asymptote (6): The line $y = L$ is a horizontal asymptote of the graph of f if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L.$

infinite limit (6): The equation $\lim_{x\to c} f(x) = \infty$ means that $f(x)$ increases without bound as x approaches c .

limit (4): If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, the limit of $f(x)$, as x approaches c, is L. We write $\lim_{x\to c} f(x) = L.$

natural logarithmic function (24): The natural logarithmic function is defined by a definite integral:

$\ln x = \int_1^x \frac{1}{t}\,dt,$ where $x > 0.$

point of inflection (14): A point $(c, f(c))$ is a point of inflection (or inflection point) if the concavity changes at that point.

point-slope equation (1): The point-slope equation of the line passing through the point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.

product rule (9, 34): The derivative of the product of 2 differentiable functions f and g is $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$. Note: The derivative of a product of 2 functions is not the product of their derivatives.

quotient rule (9): The derivative of the quotient of 2 differentiable functions f and g is $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$. Note: The derivative of a quotient of 2 functions is not the quotient of their derivatives.

slope-intercept equation of a line (2): $y = mx + b.$

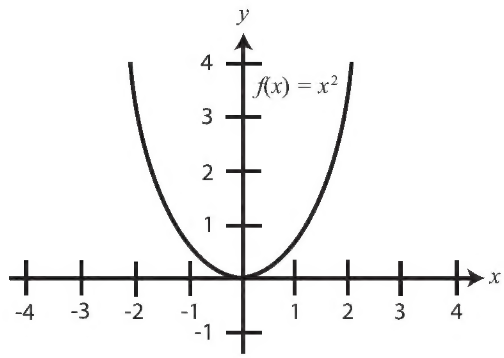
special trigonometric limits (4):

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

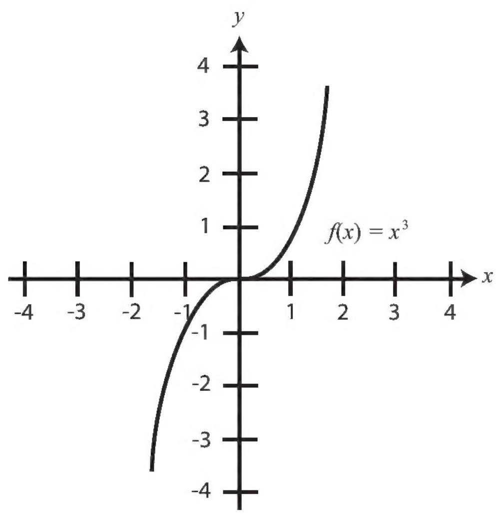
$$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0.$$

vertical asymptote (6): If $f(x)$ approaches infinity (or negative infinity) as x approaches c , from the right or left, then the vertical line $x = c$ is a vertical asymptote of the graph of f.

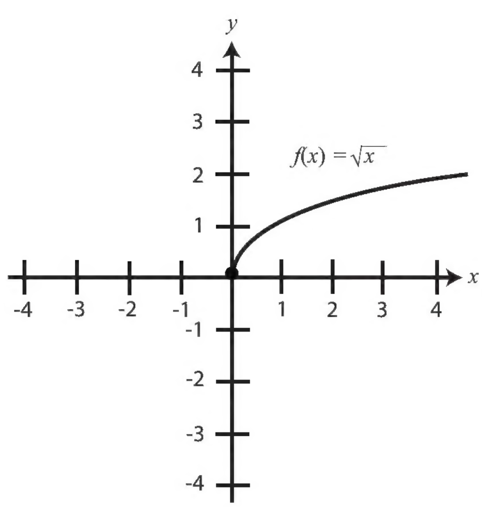
Graphs of Common Functions



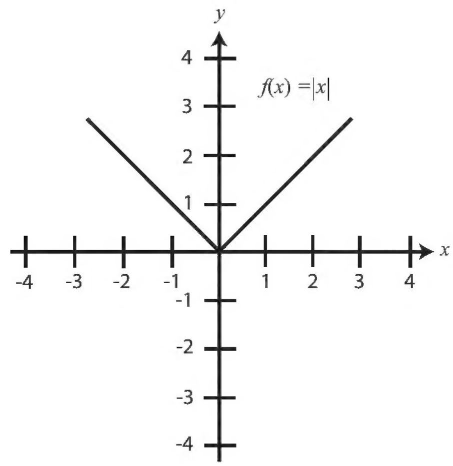
Squaring function



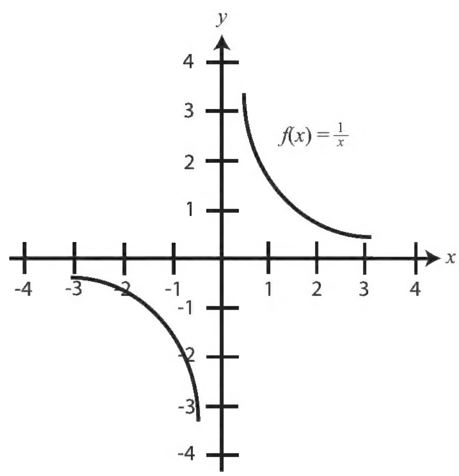
Cubing function



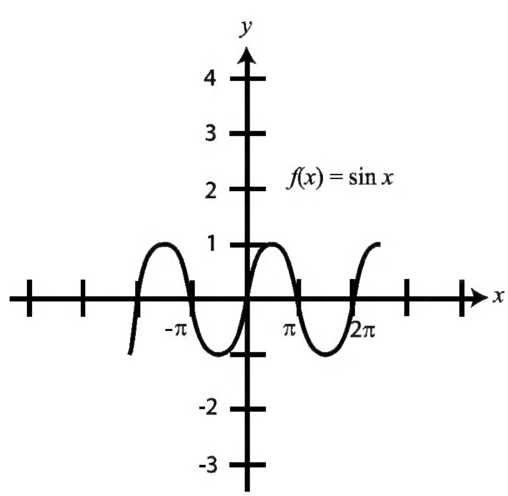
Square root function



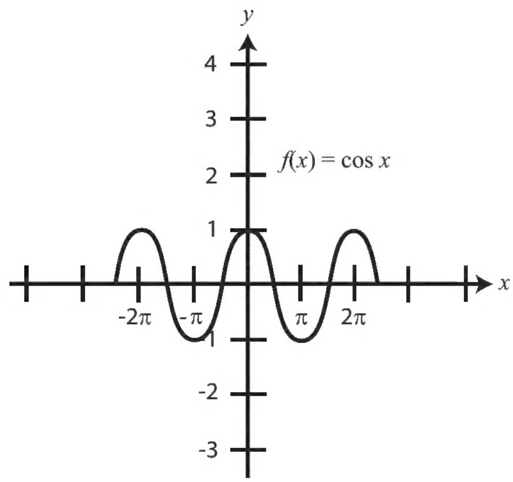
Absolute value function



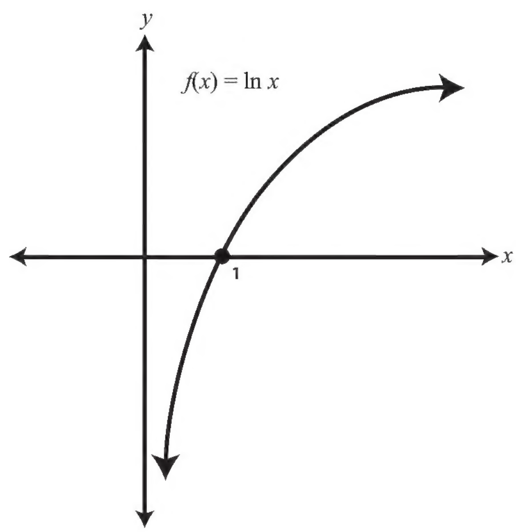
Rational function



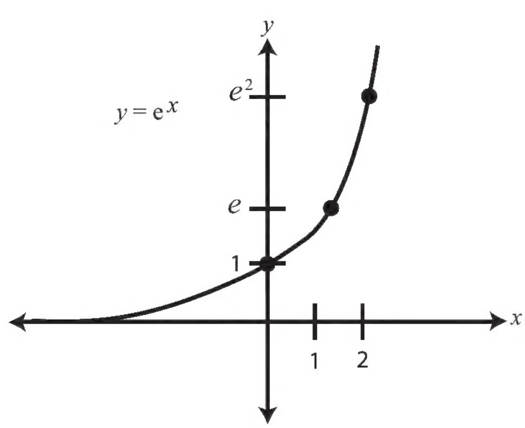
Sine function



Cosine function



Natural logarithmic function



Natural exponential function